



# On the approximation of general shell problems by the Clough-Johnson flat plate elements. Part 1 : compatibility relations

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**ON THE APPROXIMATION  
OF GENERAL SHELL PROBLEMS  
BY THE CLOUGH-JOHNSON  
FLAT PLATE ELEMENTS**

**PART 1 :  
COMPATIBILITY RELATIONS**

**Michel BERNADOU  
Yves DUCATEL**

**Mai 1987**

ON THE APPROXIMATION OF GENERAL SHELL PROBLEMSBY THE CLOUGH-JOHNSON FLAT PLATE ELEMENTSPART 1 : COMPATIBILITY RELATIONSMichel BERNADOU<sup>(1)</sup> - Yves DUCATEL<sup>(2)</sup>

Summary : In this series of reports, we give an account of some results obtained in the approximation of *general shell problems* by the CLOUGH-JOHNSON *flat plate elements*. The first part is concerned by the study of the *compatibility equations*. In the second part, we deliver several interesting results valid for general shells and we prove the "pseudo-convergence" of the method for a class of shallow shells. Then, this careful study allows us to introduce a perturbation of this approximation and thus, to propose a new method which is convergent for general shells. Finally, in the third part, we describe in details how to implement the CLOUGH-JOHNSON method.

SUR L'APPROXIMATION DE PROBLEMES GENERAUX DE COQUESPAR DES METHODES D'ELEMENTS FINIS PLATS DE CLOUGH ET JOHNSON

Résumé : Dans cette série de rapports, nous rassemblons les divers résultats obtenus dans l'approximation de *problèmes généraux de coques* par les éléments finis plats (de plaques) de CLOUGH et JOHNSON. La première partie est relative à l'étude des équations de *compatibilité*. Dans la seconde partie, nous donnons plusieurs résultats intéressants valables pour des coques générales, puis nous démontrons la pseudo-convergence de la méthode pour une classe de coques peu profondes. Cette étude détaillée nous permet alors d'introduire une perturbation de cette approximation et ainsi, de proposer une nouvelle méthode qui converge pour des coques générales. Finalement, dans la troisième partie, nous détaillons la marche à suivre pour implémenter la méthode de CLOUGH et JOHNSON.

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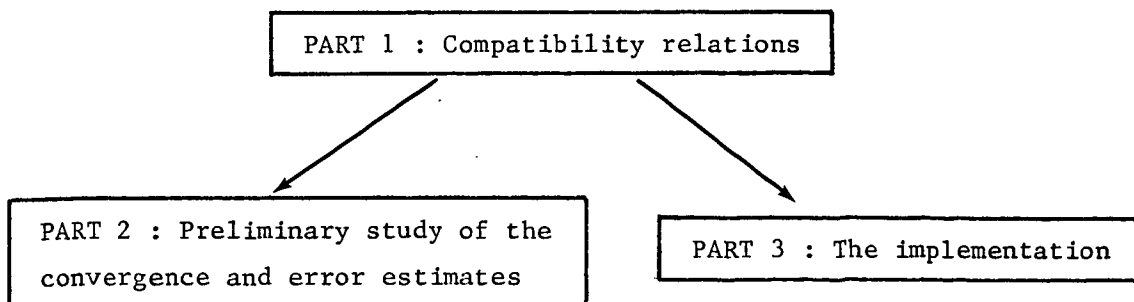
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#### HOW TO READ THE PAPER

All the paper is divided in three parts. In the first one the reader can find all the basis. Then, *part 2 and part 3 can be read independtly* depending of the interest of everyone. In other words we can summarize these possibilities in the following "flow shart" :



## 1 - INTRODUCTION

A very interesting alternative to the approximation of thin shell problems by conforming methods [1] [2] is given by the use of *flat plate elements* [3] [4]. With no doubt, such flat element methods are very popular among industrial people. These methods have as common features

- a *nonconforming approximation of the geometry* of the considered shell using facet elements ;

- a *pseudo-conforming approximation of the components of the displacement*, i.e., an approximation using conforming plate elements over every flat element.

By contrast, it seems that very few studies are available concerning *convergence and error estimate results*. Let us mention [5] [6] and [7] for the approximation of a circular arch by using straight beam elements, [8] for the approximation of a *general arch* by using the same straight beam elements, and [9][19] in the case of a portion of a right circular cylinder by using flat plate elements.

Until now, it seems that there were no results concerning the *convergence and error estimates* valid for shells whose middle surface has a *general design*. In the three following reports, we detail several interesting results obtained in this way :

*In the first one*, we give the *compatibility conditions* which have to be satisfied by the degrees of freedom at every node of the triangulation :

- continuity of the displacements at the vertices of the triangulation ;

- continuity of the tangential components (with respect to the middle surface of the shell) of the rotation vector at the vertices of the triangulation.

These compatibility conditions *should insure* the consistency between exact and approximated bilinear and linear forms. At this stage we prove that they insure the existence of *a convenient bijection* between the finite element space attached to the facet elements, constrained by the compatibility conditions, on the one hand, and the conforming finite element space attached to the initial middle surface of the shell, on the other hand.

*In the second one*, we present *several interesting results* derived when studying the convergence of the method. We prove that the compatibility conditions insure the consistency of the extensional part of the energy integrals and the consistency of the linear forms for general shells. The consistency of the flexural parts of the energy integrals depends on the form of the shell and of the respective magnitude of extensional and flexural parts. We have proved the pseudo-convergence of the CLOUGH-JOHNSON method and obtained error estimates for a class of shallow shells. Then, this careful study allows us to introduce a perturbation and thus, to define a new method which is convergent for general shells.

Finally, *in the third report*, we detail *how to implement* the CLOUGH and JOHNSON method.

Besides, we have to mention other alternatives : a facet triangular plate and shell finite element is described in [10] ; its 18 degrees of freedom are the displacement and the rotation at each corner ; this element is applicable to thin or thick shells, with or without transverse shear deformation. The previous 18 degrees of freedom element is reformulated in [12] by introducing constraints to reduce the number of such degrees to 9. Moreover, a facet element whose degrees of freedom are the displacement  $\vec{u}$  at each corner point and the normal slope  $\frac{\partial u}{\partial n}$  at the mid-points of each side is proposed in [11]. Finally, let us notice the alternative of [13] with his semi-loof shell element.

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## 2 - THE CONTINUOUS PROBLEM

In this paragraph, we give a brief description of the general linear shell model of W.T. KOITER [14] and next we recall the corresponding existence and uniqueness theorem of BERNADOU-CIARLET [15].

### 2.1 - Some notations

Let  $\mathcal{E}^3$  be the usual Euclidean space and let  $(0, \vec{e}^1, \vec{e}^2, \vec{e}^3)$  be an orthonormal reference system of  $\mathcal{E}^3$ . We denote by  $\Omega$  an open bounded subset of a plane  $\mathcal{E}^2$ , with boundary  $\Gamma$ . The *middle surface*  $\mathcal{I}$  of the shell is the image of the set  $\bar{\Omega}$  by a mapping  $\vec{\phi} : \bar{\Omega} \subset \mathcal{E}^2 \rightarrow \mathcal{E}^3$ , i.e.,

$$\mathcal{I} = \{\vec{\phi}(\xi) \text{ , } \xi = (\xi^1, \xi^2) \in \bar{\Omega}\} \text{ .}$$

Subsequently, we shall assume that

$$\vec{\phi} \in \mathcal{C}^3(\bar{\Omega}) \text{ ,} \tag{2.1.1}$$

and that at any point of the middle surface  $\mathcal{I}$  we can define the *covariant basis* (all the points of the surface  $\mathcal{I}$  are *regular*)

$$\vec{a}_1 = \vec{\phi}_{,1} \text{ , } \vec{a}_2 = \vec{\phi}_{,2} \text{ , } \vec{a}_3 = \frac{\vec{a}_1 \times \vec{a}_2}{|\vec{a}_1 \times \vec{a}_2|} \text{ .} \tag{2.1.2}$$

We shall use Greek letters for indices which take their values in the set  $\{1,2\}$ , while Latin letters will be used for indices which take their values in the set  $\{1,2,3\}$ . For these indices, we shall use Einstein's convention for summation. Finally, the notations  $f_{,\alpha}$  ,  $f_{,\alpha\beta}$  , etc..., will denote the partial derivatives  $\frac{\partial f}{\partial \xi^\alpha}$  ,  $\frac{\partial^2 f}{\partial \xi^\alpha \partial \xi^\beta}$  , etc....

The previous definitions are illustrated by Figure 2.1.1.

The *first fundamental form*  $(a_{\alpha\beta})$  of the surface  $\mathcal{I}$  is given by

$$a_{\alpha\beta} = \vec{a}_\alpha \cdot \vec{a}_\beta \text{ .} \tag{2.1.3}$$



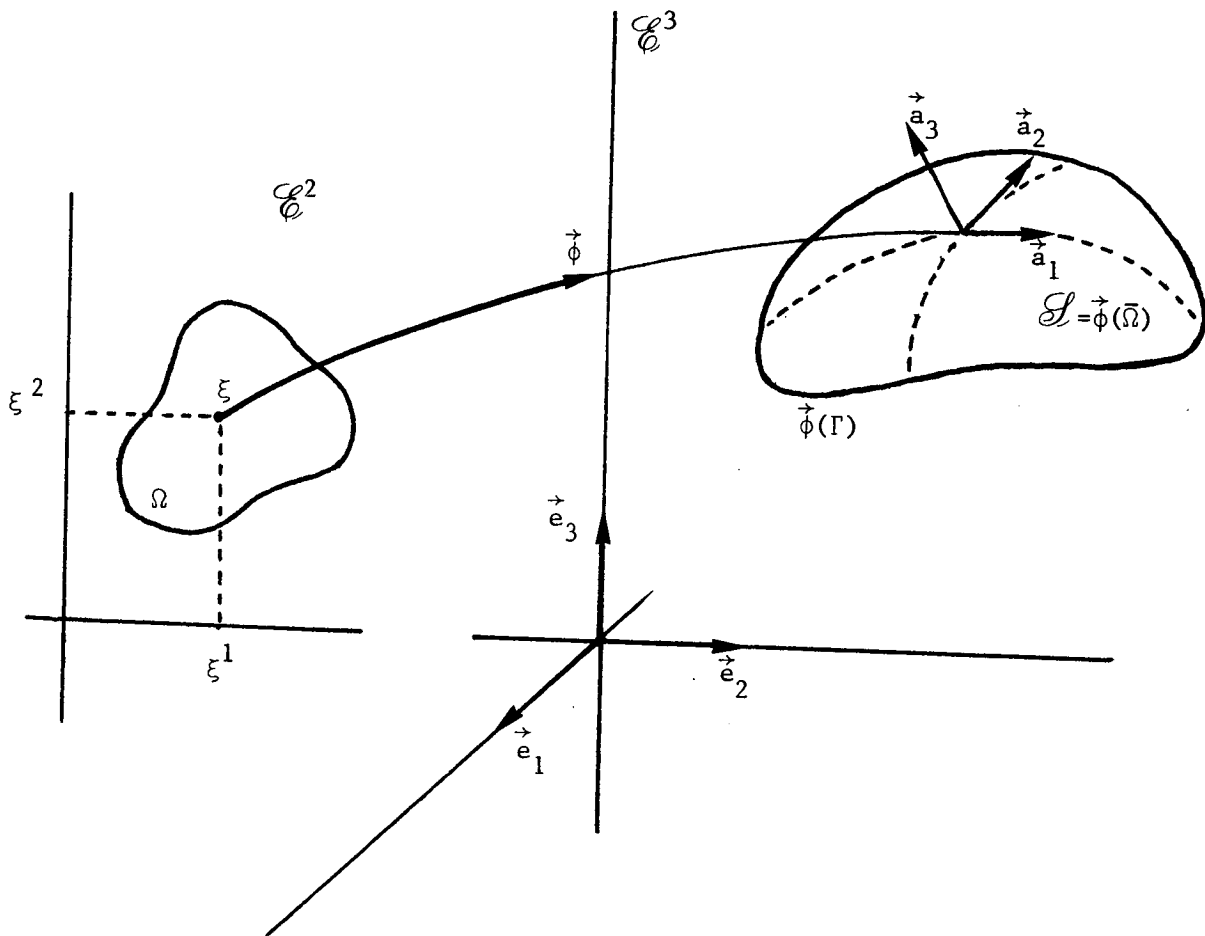


Figure 2.1.1

If we denote

$$\sqrt{a} = |\vec{a}_1 \times \vec{a}_2| , \quad (2.1.4)$$

the inverse matrix  $(a^{\alpha\beta})$  of the matrix  $(a_{\alpha\beta})$  is such that

$$a^{11} = \frac{a_{22}}{a} , \quad a^{12} = a^{21} = -\frac{a_{12}}{a} , \quad a^{22} = \frac{a_{11}}{a} . \quad (2.1.5)$$

Then, to the covariant basis  $(\vec{a}_i)$ , we associate the *contravariant basis*  $(\vec{a}^i)$  of the middle surface  $\mathcal{S}$ , i.e.,

$$\vec{a}^\alpha = a^{\alpha\beta} \vec{a}_\beta , \quad \vec{a}^3 = \vec{a}_3 . \quad (2.1.6)$$

The next expressions of some vector products will be useful :

$$\left. \begin{aligned} \vec{a}_\alpha \times \vec{a}_\beta &= \epsilon_{\alpha\beta} \vec{a}^3 , \quad \vec{a}^\alpha \times \vec{a}^\beta = \epsilon^{\alpha\beta} \vec{a}_3 , \\ \vec{a}_3 \times \vec{a}_\beta &= \epsilon_{\beta\lambda} \vec{a}^\lambda , \quad \vec{a}_3 \times \vec{a}^\beta = \epsilon^{\beta\lambda} \vec{a}_\lambda , \end{aligned} \right\} \quad (2.1.7)$$

where

$$\epsilon_{\alpha\beta} = \sqrt{a} e_{\alpha\beta} , \quad \epsilon^{\alpha\beta} = \frac{1}{\sqrt{a}} e^{\alpha\beta} , \quad (e_{\alpha\beta}) = (e^{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Moreover, by construction,

$$\vec{a}_i \cdot \vec{a}^j = \delta_i^j , \quad \delta_i^j \text{ being the Kronecker symbol.} \quad (2.1.8)$$

The *second fundamental form*  $(b_{\alpha\beta})$  of the surface is given by

$$b_{\alpha\beta} = b_{\beta\alpha} = -\vec{a}_\alpha \cdot \vec{a}_{3,\beta} = \vec{a}_3 \cdot \vec{a}_{\alpha,\beta} , \quad (2.1.9)$$

from which we derive the *third fundamental form*  $(c_{\alpha\beta})$  :

$$c_{\alpha\beta} = c_{\beta\alpha} = b_\alpha^\lambda b_{\lambda\beta} , \quad \text{where } b_\alpha^\lambda = a^{\lambda\beta} b_{\alpha\beta} . \quad (2.1.10)$$

In the following, the *unknowns of the problem* are the three functions

$$u_i : \xi \in \bar{\Omega} \rightarrow u_i(\xi) \in \mathbb{R}, \quad i = 1, 2, 3,$$

which represent the covariant components of the displacement  $\vec{u} = \vec{u}(\xi)$  of the point  $\vec{\phi}(\xi)$ , i.e.,  $\vec{u} = \sum_{i=1}^3 u_i \vec{a}^i$ .

More generally, to any displacement field  $\vec{v} = v_i \vec{a}^i$ , we associate the *strain tensor*  $(\gamma_{\alpha\beta})$  and the *change of curvature tensor*  $(\bar{\rho}_{\alpha\beta})$  through the relations

$$\gamma_{\alpha\beta}(\vec{v}) = \frac{1}{2} (v_{\beta|\alpha} + v_{\alpha|\beta}) - b_{\alpha\beta} v_3, \quad (2.1.11)$$

$$\bar{\rho}_{\alpha\beta}(\vec{v}) = v_{3|\alpha\beta} - c_{\alpha\beta} v_3 + b_{\beta|\alpha}^\lambda v_\lambda + b_{\beta}^\lambda v_{\lambda|\alpha} + b_{\alpha}^\lambda v_{\lambda|\beta}, \quad (2.1.12)$$

where the covariant derivatives are given by

$$\left. \begin{aligned} v_{\alpha|\beta} &= v_{\alpha,\beta} - \Gamma_{\alpha\beta}^\lambda v_\lambda, \\ v_{3|\alpha\beta} &= v_{3,\alpha\beta} - \Gamma_{\alpha\beta}^\lambda v_{3,\lambda}, \quad v_{3|\alpha} = v_{3,\alpha} \\ b_{\beta|\alpha}^\lambda &= b_{\beta,\alpha}^\lambda + \Gamma_{\nu\alpha}^\lambda b_{\beta}^\nu - \Gamma_{\beta\alpha}^\nu b_{\nu}^\lambda, \end{aligned} \right\} (2.1.13)$$

and

$$\Gamma_{\alpha\beta}^\lambda = \Gamma_{\beta\alpha}^\lambda = \vec{a}^\lambda \cdot \vec{a}_{\alpha,\beta}. \text{ (Christoffel symbols)} \quad (2.1.14)$$

In his analysis, KOITER [14, (4.9)] proposes the *modified change of curvature tensor*  $\rho_{\alpha\beta}$  which introduces in the energy integral an approximation of the same order than the tensor  $\bar{\rho}_{\alpha\beta}$ . The relation between both tensors is given by

$$\rho_{\alpha\beta} = \bar{\rho}_{\alpha\beta} - \frac{1}{2} (b_{\alpha}^\lambda \gamma_{\lambda\beta} + b_{\beta}^\lambda \gamma_{\lambda\alpha}). \quad (2.1.15)$$

## 2.2 - Variational formulation of KOITER's linear model

Let  $\Gamma_0$  be a measurable part of  $\Gamma$  such that  $\text{meas}(\Gamma_0) > 0$ . Subsequently, we assume that the shell is *clamped* on the part  $\partial S_0 = \vec{\phi}(\Gamma_0)$  of the middle surface boundary  $\partial S$ , i.e.,

$$\vec{u}|_{\Gamma_0} = \vec{0} \quad , \quad \partial_\nu u_3|_{\Gamma_0} = 0 \quad ,$$

where  $\partial_\nu$  denotes the outer normal derivative operator. Then, the *admissible displacement space*  $\vec{V}$  is defined by

$$\vec{V} = \{\vec{v} = (v_1, v_2, v_3) \in (H^1(\Omega))^2 \times H^2(\Omega) ; \vec{v}|_{\Gamma_0} = \vec{0} , \partial_\nu v_3|_{\Gamma_0} = 0\} \quad (2.2.1)$$

Equipped with the scalar product

$$((\vec{u}, \vec{v})) = \sum_{\alpha=1}^2 ((u_\alpha, v_\alpha))_{1,\Omega} + ((u_3, v_3))_{2,\Omega} \quad ,$$

the space  $\vec{V}$  is a Hilbert space. The corresponding norm is denoted

$$\|\vec{v}\| = [((\vec{v}, \vec{v}))]^{1/2} \quad . \quad (2.2.2)$$

For simplicity and without loss of generality, we assume that the shell is elastic, homogeneous and isotropic. Then the *strain energy of the shell* can be written

$$a(\vec{u}, \vec{v}) = \int_{\Omega} \frac{Ee}{1-\nu^2} \{ (1-\nu) \gamma_\beta^\alpha(\vec{u}) \gamma_\alpha^\beta(\vec{v}) + \nu \gamma_\alpha^\alpha(\vec{u}) \gamma_\beta^\beta(\vec{v}) + \frac{e^2}{12} [(1-\nu) \bar{\rho}_\beta^\alpha(\vec{u}) \bar{\rho}_\alpha^\beta(\vec{v}) + \nu \bar{\rho}_\alpha^\alpha(\vec{u}) \bar{\rho}_\beta^\beta(\vec{v})] \} \sqrt{a} d\xi^1 d\xi^2 \quad (2.2.3)$$

where  $\gamma_\beta^\alpha(\vec{u}) = a^{\alpha\lambda} \gamma_{\lambda\beta}(\vec{u})$  ,  $\bar{\rho}_\beta^\alpha(\vec{u}) = a^{\alpha\lambda} \bar{\rho}_{\lambda\beta}(\vec{u})$  on the one hand, and where  $e$ ,  $E$ ,  $\nu$  denote the thickness, the Young modulus and the Poisson coefficient of the shell, on the other hand.

The *potential energy of exterior forces* associated to a displacement  $\vec{v} = v_i \vec{a}^i$  of the particles of the middle surface  $\mathcal{S}$  is given by

$$f(\vec{v}) = \int_{\Omega} \vec{p} \cdot \vec{v} \sqrt{a} d\xi^1 d\xi^2 = \int_{\Omega} p^i v_i \sqrt{a} d\xi^1 d\xi^2, \quad (2.2.4)$$

where the functions  $p^i$  are the contravariant components over the basis  $(\vec{a}_i)$  of the reduced density per unit surface of the resultant over the middle surface of exterior forces applied to the shell. This expression corresponds to the case of a shell which is *free* on the complementary part  $\Gamma_1 = \Gamma - \Gamma_0$  of the boundary  $\Gamma$ .

Then, the problem of deformation of the shell  $\mathcal{S}$  in KOITER's linear model admits the following *variational formulation* : Find  $\vec{u} \in \vec{V}$  such that

$$a(\vec{u}, \vec{v}) = f(\vec{v}), \quad \forall \vec{v} \in \vec{V}, \quad (2.2.5)$$

where the forms  $a(.,.)$  and  $f(.)$  are defined by (2.2.3) and (2.2.4) respectively.

### 2.3 - Existence and uniqueness theorem

Theorem 2.3.1 : (BERNADOU-CIARLET [15, Theorem 6.4.1])

*The problem (2.2.5) has one and only one solution.*

□

## 3 - A CONFORMING FINITE ELEMENT METHOD

In this paragraph, we *only consider* an approximation of the displacement components using a conforming finite element method. This stage will be an useful intermediary for the analysis of an approximation using flat plate elements.

### 3.1 - The finite element space $\vec{V}_h$

From now on, we shall assume that the set  $\bar{\Omega}$  is a *polygon*. Then, we may *exactly* cover the set  $\bar{\Omega}$  by *regular families* of triangulations  $\mathcal{T}_h$  in the following sense :

(i) there exists a constant  $\sigma$  such that

$$\forall K \in \bigcup_h \mathcal{T}_h, \quad \frac{h_K}{\rho_K} \leq \sigma, \quad (3.1.1)$$

where  $h_K = \text{diam}(K)$  and  $\rho_K = \sup \{ \text{diam}(S), S \text{ is a ball contained in } K \}$  ;

(ii) the quantity

$$h = \max_{K \in \mathcal{T}_h} h_K \quad (3.1.2)$$

approaches zero.

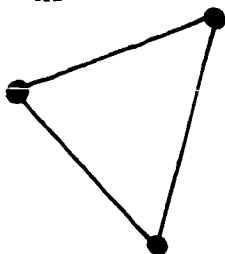
To each triangulation  $\mathcal{T}_h$  we associate a product of finite element spaces  $\vec{X}_h = X_{h1} \times X_{h1} \times X_{h2}$ . Next, we define a subspace  $\vec{V}_h = V_{h1} \times V_{h1} \times V_{h2}$  of  $\vec{X}_h$  which takes into account the boundary conditions which appear in the definition of the space  $\vec{V}$ , so that the inclusion

$$\vec{V}_h \subset \vec{V}. \quad (3.1.3)$$

holds and justifies the appellation : *conforming finite element methods*.

The different spaces are defined as follows :

Space  $X_{h1}$  is the finite element space associated to triangles of type (1),



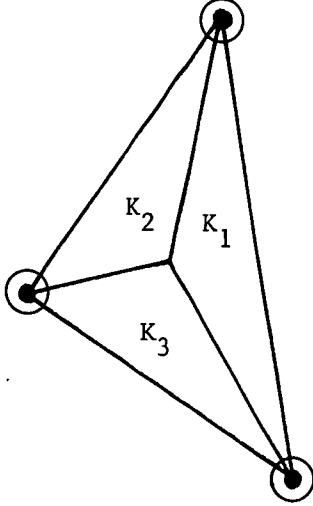
i.e., the functions of  $X_{h1}$  are such that :

- (i) on every  $K \in \mathcal{T}_h$ , they belong to  $P_1(K)$  ;
- (ii) on every  $K \in \mathcal{T}_h$ , they are determined by their values at the vertices of  $K$  ;
- (iii)  $X_{h1} \subset C^0(\bar{\Omega})$  .

(3.1.4)

Space  $V_{h1}$  : we set  $V_{h1} = \{ v_h \in X_{h1} ; v_h = 0 \text{ on } \Gamma_o \}$  .

Space  $X_{h2}$  is the finite element space associated to reduced HSIEH-



CLOUGH-TOCHER [16] triangles, i.e., the functions of  $X_{h2}$  are such that :

(i) on every  $K \in \mathcal{T}_h$ , they belong to  $P_K$ , with

$$P_K = \{p \in \mathcal{C}^1(K) ; p|_{K_i} \in P_3(K_i), 1 \leq i \leq 3,$$

$$\partial_{\nu} p|_{K'} \in P_1(K') \text{ for each side } K' \text{ of } K\} ;$$

(ii) on every  $K \in \mathcal{T}_h$ , they are determined by their values and by the values of their derivatives at the vertices of  $K$  ;

$$(iii) X_{h2} \subset \mathcal{C}^1(\bar{\Omega}).$$

(3.1.5)

Space  $V_{h2}$  : we set  $V_{h2} = \{v_h \in X_{h2} ; v_h = \partial_{\nu} v_h = 0 \text{ on } \Gamma_o\}$  .

In the previous definitions, we use the following notations :  $P_k$  is the space of all polynomials in  $\xi^1, \xi^2$  of degree  $\leq k$ . On the figures, the knowledge of the values of a function (resp. of its first derivatives) at a point is indicated by a black point (resp. a circle surrounding this point). Finally, the definitions of the spaces  $V_{h1}$  and  $V_{h2}$  suppose that the triangulation  $\mathcal{T}_h$  is realized so that  $\Gamma_o$  is an exact union of sides of triangles.

Then, the inclusions  $V_{h1} \subset V_1 = \{v | v \in H^1(\Omega), v|_{\Gamma_o} = 0\}$  and  $V_{h2} \subset V_2 = \{v | v \in H^2(\Omega), v|_{\Gamma_o} = \partial_{\nu} v|_{\Gamma_o} = 0\}$  are satisfied. Hence  $\vec{V}_h \subset \vec{V}$ .

### 3.2 - The discrete problem

The corresponding discrete problem can be stated in the plane domain as follows : Find  $\vec{u}_h \in \vec{V}_h$  such that (compare with (2.2.5)) :

$$a(\vec{u}_h, \vec{v}_h) = f(\vec{v}_h) \quad , \quad \forall \vec{v}_h \in \vec{V}_h . \quad (3.2.1)$$

The inclusion  $\vec{V}_h \subset \vec{V}$  involves the theorem :

Theorem 3.2.1 : The problem (3.2.1) has one and only one solution.

□

Using CIARLET [6], we find the following error estimate :

Theorem 3.2.2 : If the solution  $\vec{u}$  of the problem (2.2.5) belongs to the space  $\vec{V} \cap (H^2(\Omega))^2 \times H^3(\Omega)$ , then, there exists a constant C, independent of h, such that

$$\|\vec{u} - \vec{u}_h\| \leq Ch \{ |u_1|_{2,\Omega}^2 + |u_2|_{2,\Omega}^2 + |u_3|_{3,\Omega}^2 \}^{1/2} . \quad (3.2.2)$$

□

At this stage, the discrete problem is seldom realistic in practice in the sense that it would lead to exactly integrate any regular functions. Some ways to circumvent this difficulty are :

(i) to introduce a *conforming approximation of the geometry* as in CIARLET [5, § 8.2]. To be completely realistic it would need an additional approximation of the loads ;

(ii) to use *numerical integration techniques* as in BERNADOU [1] or BERNADOU-BOISSERIE [2] ;

(iii) to introduce a *nonconforming approximation of the geometry* using, or not, numerical integration.

In the following paragraphs, we shall analyze this third kind of methods, and so doing, we plan to generalize to the case of general shells, the studies of CIARLET [5, § 8.3] and Claes JOHNSON [7] related to circular arches, BERNADOU-DUCATEL [8] related to general arches, as well as BERNADOU-DUCATEL [9] and KIKUCHI [19] related to right circular cylinder.

#### 4 - THE DISCRETE PROBLEM USING FLAT PLATE ELEMENTS AND COMPATIBILITY RELATIONS

Now, we are going to define the approximate middle surface  $\bar{\mathcal{I}}_h$ . Next, we shall introduce the discrete space  $\vec{V}_h$  by using the compatibility conditions. Finally we shall set the discrete problem by using the CLOUGH-JOHNSON flat plate elements.



#### 4.1 - The approximate middle surface $\bar{\mathcal{I}}_h$

In this section we shall consider an approximation of the mapping  $\vec{\phi} = \phi^i \vec{e}_i$  defined as follows : each component  $\phi^i$ ,  $i = 1, 2, 3$ , is replaced by its interpolant  $\phi_h^i$  in the finite element space  $\Phi_h$  such that

$$\Phi_h \equiv X_{h1}, \quad X_{h1} \text{ given by (3.1.4)}. \quad (4.1.1)$$

Let  $\vec{\phi}_h \in (\Phi_h)^3$  be the interpolant of  $\vec{\phi}$ . This approximation amounts to replace the given middle surface  $\bar{\mathcal{I}}$  by a *faceted* middle surface  $\bar{\mathcal{I}}_h$ . By construction, the images of the vertices of the triangulation  $\mathcal{T}_h$  of the reference set  $\Omega$  by the mapping  $\vec{\phi}_h$  are in the initial middle surface  $\mathcal{I}$ . These considerations are illustrated by Figure 4.1.1. For more clarity, the surfaces  $\mathcal{I}$  and  $\mathcal{I}_h$  have been separated and the domain  $\Omega$  is taken rectangular.

To each flat triangle  $k = \vec{\phi}_h(K)$ ,  $K \in \mathcal{T}_h$ , we are able to associate, by analogy with section 2.1, local basis, fundamental forms, Christoffel symbols, i.e., with obvious notations (we will check in § 5.2 that  $a_h = a + O(h)$  so that  $\vec{a}_{h1} \times \vec{a}_{h2} \neq \vec{0}$  for  $h$  sufficiently small) :

$$\vec{a}_{h\alpha} = \vec{\phi}_{h,\alpha}, \quad \vec{a}_{h3} = \frac{\vec{a}_{h1} \times \vec{a}_{h2}}{|\vec{a}_{h1} \times \vec{a}_{h2}|}, \quad (4.1.2)$$

$$\sqrt{a_h} = |\vec{a}_{h1} \times \vec{a}_{h2}|, \quad (4.1.3)$$

$$a_{h\alpha\beta} = \vec{a}_{h\alpha} \cdot \vec{a}_{h\beta}, \quad (4.1.4)$$

$$\vec{a}_h^\alpha = a_h^{\alpha\beta} \vec{a}_{h\beta}, \quad \vec{a}_h^3 = \vec{a}_{h3}, \quad (4.1.5)$$

where the matrix  $(a_h^{\alpha\beta})$  is the inverse of the matrix  $(a_{h\alpha\beta})$ ,

$$b_{h\alpha\beta} = b_{h\alpha}^\beta = c_{h\alpha\beta} = \Gamma_{h\alpha\beta}^\lambda = 0. \quad (4.1.6)$$

It is worth to note that all the previous quantities are *constant* in any given triangle  $K \in \mathcal{T}_h$  with possible discontinuities on the interfaces due to the discontinuity of the first derivatives of  $\vec{\phi}_h$  on these interfaces.

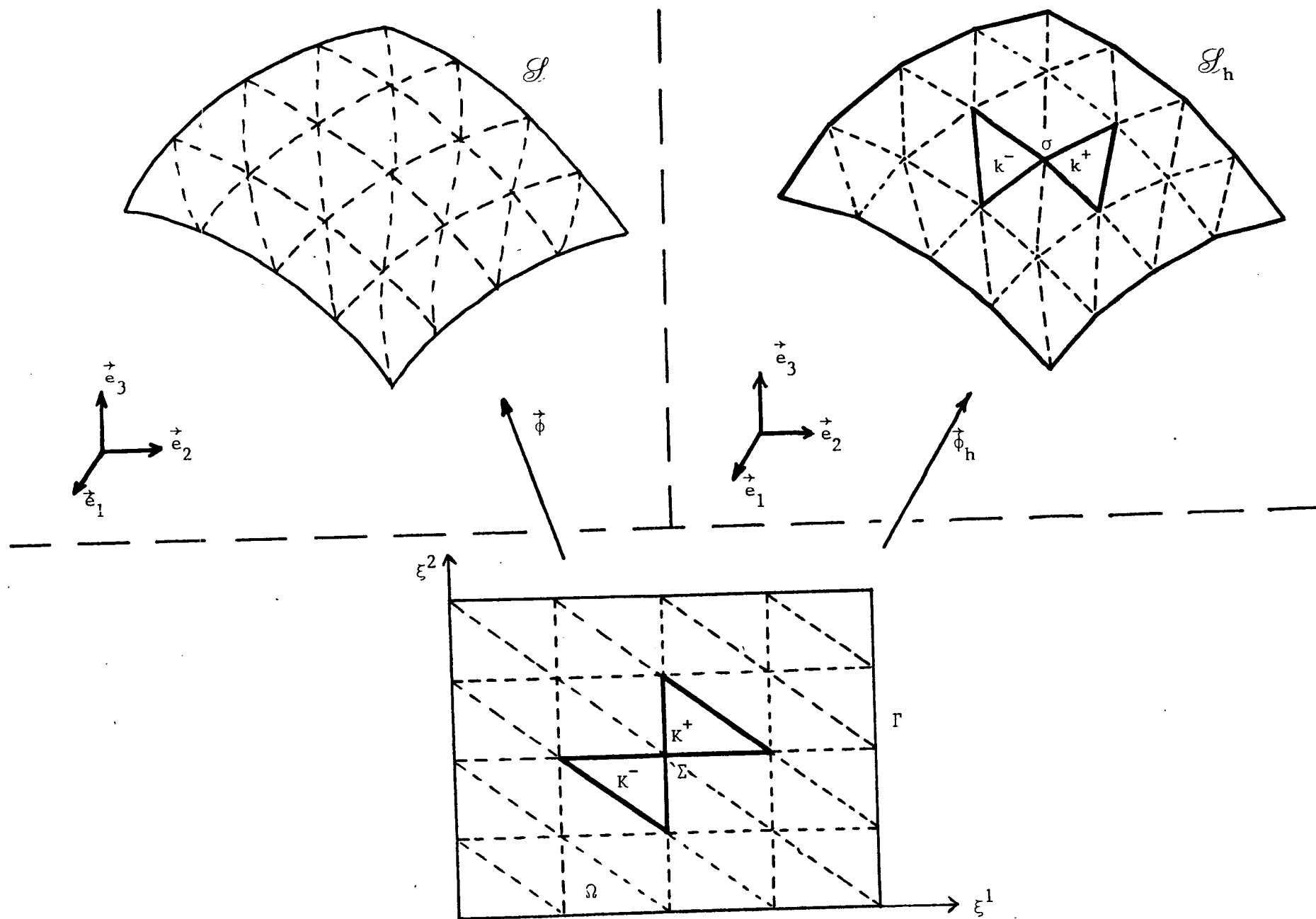


Figure 4.1.1 : The initial middle surface  $\mathcal{S}$  and the faceted middle surface  $\mathcal{S}_h$

#### 4.2 - The discrete space $\vec{V}_h$ using compatibility conditions

One of the main ideas of the approximation of general shell problems by flat plate elements is to approach the energy of the shell by a sum of plate energies defined on each facet of the approximated middle surface  $\mathcal{I}_h$ .

In this goal, we shall introduce in a first step a discrete space  $\vec{X}_h$ , i.e.,

$$\vec{X}_h = \vec{X}_{h1} \times \vec{X}_{h1} \times \vec{X}_{h2} . \quad (4.2.1)$$

The space  $\vec{X}_{h1}$  (resp.  $\vec{X}_{h2}$ ) will be used to approximate the components  $\vec{v}_{h1}$  and  $\vec{v}_{h2}$  (resp. the component  $\vec{v}_{h3}$ ) on the local basis  $\vec{a}_{h1}$  (see (4.1.2)) of a displacement field  $\vec{v}_h$ . Since the local basis are constant in any triangle  $K \in \mathcal{T}_h$  with possible discontinuities on the interfaces, the functions of the spaces  $\vec{X}_{h\alpha}$  will be determined triangle by triangle without imposing any connection between functions defined on adjacent triangles.

In a second step, in order to get an approximate energy which is consistent to the energy of the shell, we shall introduce constraints, i.e. compatibility conditions, on the functions of the space  $\vec{X}_h$ . The corresponding subspace  $\vec{X}_h$  will be

$$\vec{X}_h = \{ \vec{v}_h \in \vec{X}_h ; \vec{v}_h \text{ satisfies the compatibility conditions} \} . \quad (4.2.2)$$

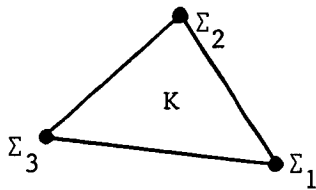
Next, in the theorem 4.2.1 we shall prove the existence of a "suitable" bijection  $F_h$  between the space  $\vec{X}_h$  and the space  $\vec{X}_h$  considered in paragraph 3.1.

Finally, in a third step, we shall take into account the boundary conditions and thus, we shall get the space  $\vec{V}_h$ .

Definition of the space  $\tilde{X}_h$  :

According to relation (4.2.1) we have to define the discrete spaces  $\tilde{X}_{h1}$  and  $\tilde{X}_{h2}$  :

Space  $\tilde{X}_{h1}$  : The functions of the space  $\tilde{X}_{h1}$  are independently defined triangle by triangle. On a given triangle  $K \in \mathcal{T}_h$  with vertices  $\Sigma_1, \Sigma_2, \Sigma_3$ , the restriction  $v_h|_K$  of any function  $v_h \in \tilde{X}_{h1}$  is such that

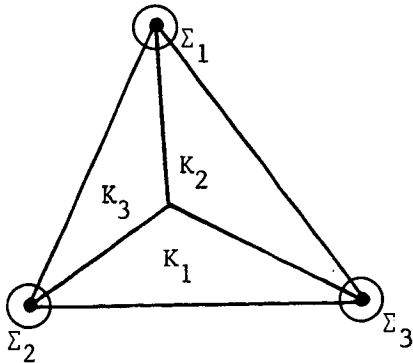


- (i)  $v_h|_K \in P_1(K)$  ;
- (ii)  $v_h|_K$  is completely specified by the values  $v_h|_K(\Sigma_i)$ ,  $i = 1, 2, 3$  .

(4.2.3)

Next, the definition of the space  $\tilde{X}_{h2}$  uses the reduced HSIEH-CLOUGH-TOCHER [16] triangle :

Space  $\tilde{X}_{h2}$  : The functions of this space are independently defined triangle by triangle. On a given triangle  $K \in \mathcal{T}_h$  partitioned in three nondegenerated triangles  $K_i$ ,  $i = 1, 2, 3$ , the restriction  $v_h|_K$  of any function  $v_h \in \tilde{X}_{h2}$  is such that :



- (i)  $v_h|_K \in P_K$  , with  
 $P_K = \{p \in \mathcal{C}^1(K) ; p|_{K_i} \in P_3(K_i), 1 \leq i \leq 3, \partial_\nu p|_{K'} \in P_1(K') \text{ for each side } K' \text{ of } K\}$  ;
- (ii)  $v_h|_K$  is completely determined by the values  $v_h|_K(\Sigma_i), Dv_h|_K(\Sigma_i)$  ,  $i = 1, 2, 3$  .

(4.2.4)

These definitions of the spaces  $\tilde{X}_{h1}$  and  $\tilde{X}_{h2}$  involve immediately that the space  $\tilde{X}_h$  has  $15 M_h$  degrees of freedom, where  $M_h$  denotes the number of triangles of the triangulation.

Derivation of the compatibility conditions :

To derive the compatibility conditions it is convenient to consider two facets  $k^+$  and  $k^-$  of the approximate middle surface  $\mathcal{S}_h$  which have a common vertex  $\sigma$  (see Figure 4.1.1). This common vertex  $\sigma$  can be regarded as

(i) a point of the middle surface  $\mathcal{S}$  : then, using the notations of section 2.1, any displacement field  $\vec{v} \in \vec{V}$  of the middle surface  $\mathcal{S}$  has the following components at point  $\sigma$ , or, more conveniently, at point  $\Sigma$  (using the mapping  $\vec{\phi}$ ) :

$$\vec{v}(\Sigma) = v_i(\Sigma) \vec{a}^i(\Sigma) . \quad (4.2.5)$$

To the displacement field  $\vec{v}$  we associate the usual rotation vector  $\vec{\omega}$ , the strain tensor  $\gamma_{\alpha\beta}$  and the change of curvature tensor  $\bar{\rho}_{\alpha\beta}$  which expressions are given by (see KOITER [14]) :

$$\begin{aligned} \vec{\omega}(\vec{v}) &= \omega^i(\vec{v}) \vec{a}_i , \\ \text{with } \omega^\lambda(\vec{v}) &= \varepsilon^{\lambda\beta} (v_{3|\beta} + b_\beta^\alpha v_\alpha) , \quad \omega^3(\vec{v}) = \frac{1}{2} \varepsilon^{\lambda\beta} v_{\beta|\lambda} , \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} (4.2.6)$$

$$\gamma_{\alpha\beta}(\vec{v}) = \frac{1}{2} (v_{\beta|\alpha} + v_{\alpha|\beta}) - b_{\alpha\beta} v_3 , \quad (4.2.7)$$

$$\bar{\rho}_{\alpha\beta}(\vec{v}) = v_{3|\alpha\beta} - c_{\alpha\beta} v_3 + b_\beta^\lambda |_\alpha v_\lambda + b_\beta^\lambda v_{\lambda|\alpha} + b_\alpha^\lambda v_{\lambda|\beta} \quad (4.2.8)$$

The previous expressions can be attached to *any* displacement field  $\vec{v}$  of the middle surface  $\mathcal{S}$ , particularly to the solution  $\vec{u}$  of the problem (2.2.5) and to the solution  $\vec{u}_h$  of the discrete problem (3.2.1).

(ii) a vertex of the facet  $k^+ = \vec{\phi}_h(K^+)$ . For convenience, we shall sometimes use the superscript  $+$  to refer to parameters attached to the triangle  $K^+$ . Thus we shall write  $\Sigma^+$  instead of  $\Sigma$ ,  $\vec{a}_h^{i+}$  instead of  $\vec{a}_h^i$ , etc. Then, any displacement field  $\vec{v}_h \in \vec{X}_h$  of the faceted surface  $\mathcal{S}_h$  has the following components at point  $\sigma^+ = \sigma$  (or, using  $\vec{\phi}_h$  at point  $\Sigma^+$ ) :

$$\vec{\tilde{v}}_h(\Sigma^+) = \vec{\tilde{v}}_{hi}(\Sigma^+) \vec{a}_h^{i+} . \quad (4.2.9)$$

To the displacement field  $\vec{\tilde{v}}_h$ , we associate the usual rotation vector  $\vec{\tilde{\omega}}_h$ , the strain tensor  $\vec{\tilde{\gamma}}_{h\alpha\beta}$  and the change of curvature tensor  $\vec{\tilde{\rho}}_{h\alpha\beta}$  which expressions are obtained triangle by triangle using the mapping  $\vec{\phi}_h$ . By using relation (4.1.6) and by denoting  $\vec{\tilde{v}}_h^+ = \vec{\tilde{v}}_h|_{K^+}$ , we obtain for the triangle  $K^+$  :

$$\left. \begin{aligned} \vec{\tilde{\omega}}_h(\vec{\tilde{v}}_h^+) &= \vec{\tilde{\omega}}_h^{i+} \vec{a}_{hi}^{++} , \\ \text{with } \vec{\tilde{\omega}}_h^{\lambda+} &= \frac{1}{\sqrt{a_h^+}} e^{\lambda\beta} \vec{\tilde{v}}_{h3,\beta}^+ , \quad \vec{\tilde{\omega}}_h^{3+} = \frac{1}{2\sqrt{a_h^+}} e^{\lambda\beta} \vec{\tilde{v}}_{h\beta,\lambda}^+ \end{aligned} \right\} (4.2.10)$$

$$\vec{\tilde{\gamma}}_{h\alpha\beta}(\vec{\tilde{v}}_h^+) = \frac{1}{2} (\vec{\tilde{v}}_{h\beta,\alpha}^+ + \vec{\tilde{v}}_{h\alpha,\beta}^+) , \quad \vec{\tilde{\gamma}}_{h\beta}^\alpha(\vec{\tilde{v}}_h^+) = a_h^{\alpha\lambda+} \vec{\tilde{\gamma}}_{h\lambda\beta}(\vec{\tilde{v}}_h^+) , \quad (4.2.11)$$

$$\vec{\tilde{\rho}}_{h\alpha\beta}(\vec{\tilde{v}}_h^+) = \vec{\tilde{v}}_{h3,\alpha\beta}^+ , \quad \vec{\tilde{\rho}}_{h\beta}^\alpha(\vec{\tilde{v}}_h^+) = a_h^{\alpha\lambda+} \vec{\tilde{\rho}}_{h\lambda\beta}(\vec{\tilde{v}}_h^+) ; \quad (4.2.12)$$

(iii) a vertex of the facet  $k^- = \vec{\phi}_h(K^-)$ . We can derive similar results than in (ii) ; it suffices to replace the superscript + by the superscript - .

Now, we are able to get the *compatibility conditions* which consist in constraints imposed to the functions  $\vec{\tilde{v}}_h$  of the space  $\vec{X}_h$ , i.e., they correspond to relations that the degrees of freedom of the space  $\vec{X}_h$  have to satisfy. We derive them by writting that :

(i) the displacement  $\vec{\tilde{v}}_h$  is continuous at the vertices  $\sigma$  of the surface  $\mathcal{I}$ , or equivalently, at the vertices  $\Sigma$  of the triangulation  $\mathcal{T}_h$ , i.e.,

$$\vec{\tilde{v}}_h(\Sigma^+) = \vec{\tilde{v}}_h(\Sigma^-) , \quad \forall \Sigma \text{ vertex of } \mathcal{T}_h . \quad (4.2.13)$$

Next, we remark that the knowledge of the values of the degrees of freedom of the function  $\vec{\tilde{v}}_h \in \vec{X}_h$  involves the knowledge of the values of the rotation vectors  $\vec{\tilde{\omega}}_h(\Sigma^+)$  and  $\vec{\tilde{\omega}}_h(\Sigma^-)$  for any vertex  $\Sigma$  of  $\mathcal{T}_h$  .

Indeed, using the relations (4.2.10), the result is immediate for the components  $\tilde{\omega}_h^{\lambda+}$ . For the third component, we have just to remember that  $\tilde{v}_{h\beta}^+ \in P_1(K^+)$ ; hence the values  $\tilde{v}_{h\beta,\lambda}^+(\Sigma^+)$  are just depending on the values of  $\tilde{v}_{h\beta}^+$  at the three vertices of the triangle  $K^+$ . Obviously, this argument can be applied in the same way to prove that  $\tilde{\omega}_h(\Sigma^-)$  is well known. Hence we can state the second kind of conditions :

(ii) the tangential components - with respect to the middle surface  $\mathcal{S}$  - of the rotation vector  $\tilde{\omega}_h(\tilde{v}_h)$  are continuous at the vertices  $\sigma$  of the surface  $\mathcal{S}$  (or  $\mathcal{S}_h$ ) or equivalently at the vertices  $\Sigma$  of  $\mathcal{C}_h$ , i.e.,

$$\tilde{\omega}_h(\Sigma^+).a^\alpha(\Sigma) = \tilde{\omega}_h(\Sigma^-).a^\alpha(\Sigma) , \quad \forall \Sigma \text{ vertex of } \mathcal{C}_h . \quad (4.2.14)$$

These conditions seem to be natural enough from mechanical viewpoint. They correspond to the conditions used by CLOUGH-JOHNSON [3,4]. Moreover, they generalize the compatibility conditions considered by CIARLET [5, § 8.3] and Claes JOHNSON [7] in case of circular arches, by BERNADOU-DUCATEL [8] in case of general arches and by BERNADOU-DUCATEL [9] and KIKUCHI [19] in case of rigid circular cylinder.

Let  $\tilde{X}_h$  be the following space

$$\tilde{X}_h = \{ \tilde{v}_h \in \tilde{X}_h ; \tilde{v}_h \text{ satisfies the compatibility conditions} \} \quad (4.2.13) \quad (4.2.14) \quad (4.2.15)$$

and let us state the following hypothesis :

Hypothesis 4.2.1 : Let  $\sigma = \sigma^+ = \sigma^-$  be a vertex of the faceted surface  $\mathcal{S}_h$  respectively considered as a point of  $\mathcal{S}$ , as a vertex of a facet  $k^+$  and as a vertex of a facet  $k^-$ .

Then, by using the correspondence  $\sigma = \vec{\phi}(\Sigma)$  and by denoting  $\vec{a}^3(\Sigma)$ ,  $\vec{a}_h^{3+}$ ,  $\vec{a}_h^{3-}$  the corresponding normal vector to the middle surface  $\mathcal{S}$ , to the facet  $k^+$  and to the facet  $k^-$ , we assume

$$\vec{a}^3(\Sigma) \cdot \vec{a}_h^{3+} \neq 0, \vec{a}^3(\Sigma) \cdot \vec{a}_h^{3-} \neq 0, \forall \Sigma \text{ vertex of two adjacent triangles } K^+ \text{ and } K^-. \quad (4.2.16)$$

□

It can be proved that the assumptions:  $\vec{\phi} \in \mathcal{C}^3(\Omega)$ , and all the points of  $\mathcal{S}$  are regular, imply the existence of a constant  $h_1$  such that the relations (4.2.16) are satisfied  $\forall h < h_1$ . But it is interesting to note that if we had had to consider a junction between two different surfaces  $\mathcal{S}_1$  and  $\mathcal{S}_2$  with discontinuity of the tangent planes along the junction, the previous result would not have been always true.

Now we are going to prove the following theorem which will be very useful for the next purposes :

Theorem 4.2.1 : Under the hypothesis 4.2.1, there exists a bijection  $F_h$  between the spaces  $\vec{X}_h$  and  $\vec{X}_h$ .

Proof :

$$(i) \vec{v}_h \in \vec{X}_h \rightarrow \vec{v}_h \in \vec{X}_h :$$

Let us consider any function  $\vec{v}_h \in \vec{X}_h$ , i.e.,  $\vec{v}_h \in \vec{X}_h$  and  $\vec{v}_h$  satisfies the relations (4.2.13) (4.2.14). Then, to the set of values of the degrees of freedom of the function  $\vec{v}_h$  we associate the following set of values :

$$\vec{A}(\Sigma) = \vec{v}_h(\Sigma^+) = \vec{v}_h(\Sigma^-), \forall \Sigma \text{ vertex of } \mathcal{T}_h, \quad (4.2.17)$$

$$B^\alpha(\Sigma) = \vec{\omega}_h(\Sigma^+) \cdot \vec{a}^\alpha(\Sigma) = \vec{\omega}_h(\Sigma^-) \cdot \vec{a}^\alpha(\Sigma), \forall \Sigma \text{ vertex of } \mathcal{T}_h. \quad (4.2.18)$$

And now, to the set of values (4.2.17)(4.2.18) we associate the function  $\vec{v}_h \in \vec{X}_h$  determined by the following values of its degrees of freedom :



$$\vec{v}_h(\Sigma) = \vec{A}(\Sigma) \quad , \quad \forall \Sigma \text{ vertex of } \mathcal{T}_h \quad , \quad (4.2.19)$$

$$v_{h3,v}(\Sigma) = \varepsilon_{\alpha v}(\Sigma) B^\alpha(\Sigma) - b_v^\alpha(\Sigma) \vec{A}(\Sigma) \cdot \vec{a}_\alpha(\Sigma) \quad , \quad \forall \Sigma \text{ vertex of } \mathcal{T}_h \quad . \quad (4.2.20)$$

The equation (4.2.20) requires some explanations : the relations (2.1.13) (4.2.6) and the relation  $\varepsilon_{\alpha v} \varepsilon^{\alpha \lambda} = \delta_v^\lambda$  (see (2.1.7)) involve

$$v_{3,v} = \varepsilon_{\alpha v} \omega^\alpha(\vec{v}) - b_v^\alpha v_\alpha \quad ,$$

and, moreover, we use the relation  $v_\alpha = \vec{v} \cdot \vec{a}_\alpha$  .

Thus, to any given function  $\vec{v}_h \in \vec{X}_h$ , the relations (4.2.17)-(4.2.20) associate one and only one function  $\vec{v}_h \in \vec{X}_h$ .

Now, we consider the converse property :

$$(ii) \quad \vec{v}_h \in \vec{X}_h \rightarrow \vec{v}_h \in \vec{X}_h \quad :$$

The knowledge of  $\vec{v}_h \in \vec{X}_h$  is equivalent to the knowledge of its degrees of freedom  $\vec{v}_h(\Sigma)$  and  $v_{h3,v}(\Sigma)$  for all vertices  $\Sigma$  of  $\mathcal{T}_h$ . To this set of values we associate the following set of values :

$$\vec{A}(\Sigma) = \vec{v}_h(\Sigma) \quad , \quad \forall \Sigma \text{ vertex of } \mathcal{T}_h \quad , \quad (4.2.21)$$

$$B^\lambda(\Sigma) = \varepsilon^{\lambda \beta}(\Sigma) [v_{h3,\beta}(\Sigma) + b_\beta^\alpha(\Sigma) v_{h\alpha}(\Sigma)] \quad , \quad \forall \Sigma \text{ vertex of } \mathcal{T}_h \quad . \quad (4.2.22)$$

Then, we take the following values for the first part of the set of degrees of freedom of the searched function  $\vec{v}_h \in \vec{X}_h$  :

$$\vec{v}_h(\Sigma^+) = \vec{v}_h(\Sigma^-) = \vec{A}(\Sigma) \quad . \quad (4.2.23)$$

From these values we can derive the values of

$$\tilde{\omega}_h^3(\Sigma^+) = \frac{1}{2\sqrt{a_h^+}} e^{\lambda \beta} \tilde{v}_{h\beta,\lambda}^+(\Sigma^+) \quad , \quad \tilde{\omega}_h^3(\Sigma^-) = \frac{1}{2\sqrt{a_h^-}} e^{\lambda \beta} \tilde{v}_{h\beta,\lambda}^-(\Sigma^-) \quad , \quad (4.2.24)$$

by applying (4.2.10) and by using the analysis which follows the relation (4.2.13).

Then, we are able to determine the components  $\tilde{\omega}_h^\beta(\Sigma^+)$  and  $\tilde{\omega}_h^\beta(\Sigma^-)$  by writting that the vectors  $\vec{z}_h(\Sigma^+) = \tilde{\omega}_h^i(\Sigma^+) \vec{a}_{hi}^+$  and  $\vec{z}_h(\Sigma^-) = \tilde{\omega}_h^i(\Sigma^-) \vec{a}_{hi}^-$  are the unique solutions of the system constituted by the equations

$$\vec{z}_h(\Sigma^+) \cdot \vec{a}^\lambda(\Sigma) = \vec{z}_h(\Sigma^-) \cdot \vec{a}^\lambda(\Sigma) = B^\lambda(\Sigma), \quad \forall \Sigma \text{ vertex of } \mathcal{C}_h. \quad (4.2.25)$$

Above equations (4.2.25)<sup>+</sup> can be written in the form

$$\tilde{\omega}_h^\beta(\Sigma^+) \vec{a}_{h\beta}^+ \cdot \vec{a}^\lambda(\Sigma) = B^\lambda(\Sigma) - \tilde{\omega}_h^3(\Sigma^+) \vec{a}_{h3}^+ \cdot \vec{a}^\lambda(\Sigma).$$

These equations permit to derive  $\tilde{\omega}_h^\beta(\Sigma^+)$  if we observe that  $\det(\vec{a}_{h\beta}^+ \cdot \vec{a}^\lambda(\Sigma)) \neq 0$ . To prove this last statement we notice that  $\vec{a}^\lambda(\Sigma) = (\vec{a}_{hi}^+ \cdot \vec{a}^\lambda(\Sigma)) \vec{a}_h^{i+}$  and thus, on the one hand

$$(\vec{a}^1(\Sigma) \times \vec{a}^2(\Sigma)) \cdot \vec{a}_h^{3+} = \frac{\vec{a}^3(\Sigma) \cdot \vec{a}_h^{3+}}{\sqrt{a(\Sigma)}}$$

and, on the other hand,

$$(\vec{a}^1(\Sigma) \times \vec{a}^2(\Sigma)) \cdot \vec{a}_h^{3+} = \frac{1}{\sqrt{a_h^+}} \det(\vec{a}_{h\beta}^+ \cdot \vec{a}^\lambda(\Sigma)).$$

Then, the result arises from the relation (4.2.16) of the hypothesis 4.2.1. Exactly in the same way, we can derive the components  $\tilde{\omega}_h^\beta(\Sigma^-)$ .

Next, from  $\tilde{\omega}_h^\beta(\Sigma^+)$ ,  $\tilde{\omega}_h^\beta(\Sigma^-)$  and relations (4.2.10), we get

$$\tilde{v}_{h3,\beta}(\Sigma^+) = e_{\lambda\beta} \sqrt{a_h^+} \tilde{\omega}_h^\lambda(\Sigma^+), \quad \tilde{v}_{h3,\beta}(\Sigma^-) = e_{\lambda\beta} \sqrt{a_h^-} \tilde{\omega}_h^\lambda(\Sigma^-). \quad (4.2.26)$$

Then, the relations (4.2.23) (4.2.26) determine one and only one function  $\vec{z}_h \in \tilde{X}_h$ . Since by definition this function satisfies the relations (4.2.23) (4.2.25), i.e., the compatibility relations, the function  $\vec{z}_h$  belongs in fact to the subspace  $\tilde{X}_h$ .

Finally, one can check easily that the application of step (ii) is, by construction, the inverse  $F_h^{-1}$  of the application  $F_h$  defined in section (i).

□

Geometrical aspects of the functions of the space  $\vec{X}_h$  :

Let us consider any function  $\vec{v}_h \in \vec{X}_h$ . By definition  $\vec{v}_h = \tilde{v}_{hi} \vec{a}_h^i$ ,  $\tilde{v}_{h1} \in \tilde{X}_{h1}$ ,  $\tilde{v}_{h2} \in \tilde{X}_{h1}$ ,  $\tilde{v}_{h3} \in \tilde{X}_{h2}$ . Let us refer the functions  $\tilde{v}_{hi}$ , defined on each triangle  $K \in \mathcal{T}_h$ , to the normal to the domain  $\Omega$ . We get

(i) for the functions  $\tilde{v}_{h1}$  and  $\tilde{v}_{h2}$ , a *discontinuous plane faceted surface*. This proceeds from the fact that  $\tilde{v}_{hi}|_K \in P_1(K)$  and that the components  $\tilde{v}_{hi}$  are taken with respect to basis  $\vec{a}_h^i$  which are constant triangle by triangle and generally *discontinuous* from one triangle to the other.

(ii) for the function  $\tilde{v}_{h3}$ , a *discontinuous curved faceted surface*. One can see this by analogy with (i), but now  $\tilde{v}_{h3}|_K \in P_K$ ,  $P_K$  defined in (4.2.4).

Finally, we observe that the displacement field  $\vec{v}_h \in \vec{X}_h$  associates to the faceted middle surface  $\mathcal{I}_h$  (which is an approximation of the undeformed middle surface) a *discontinuous curved faceted surface*  $\mathcal{I}_h^*$ . Let us note that by definition of the space  $\vec{X}_h$ , the surface  $\mathcal{I}_h^*$  is continuous at the vertices  $\Sigma$  of the triangulation  $\mathcal{T}_h$ . In other words, the surface  $\mathcal{I}_h$  looks like some modern parachutes.

Thus, in general we do not have  $\vec{X}_h \subset (H^1(\Omega))^2 \times H^2(\Omega)$  but only,

$$\vec{X}_h \subset \prod_{K \in \mathcal{T}_h} (H^1(K) \times H^1(K) \times H^2(K)) .$$

At this stage one can wonder why to use  $\mathcal{C}^1$ -element to approach  $\tilde{v}_3$  since we do not have  $\tilde{X}_{h3} \subset \mathcal{C}^1(\bar{\Omega})$ . This is essentially to realize a conforming approximation of plate equations on every facet and to find again the conforming finite element method for plates in the particular case where the middle surface  $\mathcal{I}$  of the shell is plane.

Of course, let us add that, by construction, the corresponding functions  $\vec{v}_h = F_h(\vec{v}_h)$ ,  $\forall \vec{v}_h \in \vec{X}_h$ , are such that

$$\vec{v}_h \in (\mathcal{C}^0(\bar{\Omega}))^2 \times \mathcal{C}^1(\bar{\Omega}) .$$

Remark 4.2.1 : The choice of the equations (4.2.14) among the set of *compatibility conditions* calls the two following comments :

(i) these conditions state the *continuity of the components* of the rotation vector on the *tangent plane to the surface  $\mathcal{S}$* . Particularly, these conditions specify *nothing* on the normal component of the rotation vector. Let us note that the component  $\omega^3(\vec{v}_h)$ ,  $\vec{v}_h \in \vec{X}_h$ , is defined triangle by triangle and is generally discontinuous at the vertices  $\Sigma$  and on the interfaces.

(ii) the use of the geometry of the middle surface requires the knowledge of the analytical definition of  $\mathcal{S}$ . This situation often occurs practically. However, it would be interesting to generalize this flat plate element method to the case of a description of the middle surface by the data of the coordinates of a finite number of its points (for instance, the vertices  $\sigma$  of the faceted middle surface) and, eventually, by the data of the tangent plane to the middle surface at these points. In this way see BERNADOU-LALANNE [17].

□

Definition of the space  $\vec{V}_h$

To study the convergence of the flat plate element method it is useful to define a subspace  $\vec{V}_h$  of the space  $\vec{X}_h$  which image, through the bijection  $F_h$ , is precisely the space  $\vec{V}_h$  defined by (3.1.3) (3.1.4) (3.1.5), i.e.,

$$\vec{V}_h = \{ \vec{v}_h \in \vec{X}_h ; \vec{v}_h|_{\Gamma_o} = \vec{0} ; \partial_v v_{h3} = 0 \text{ on } \Gamma_o \} . \quad (4.2.27)$$

According to the definition of the space  $\vec{X}_h$ , to get the boundary conditions  $\vec{v}_h|_{\Gamma_o} = \vec{0}$  and  $\partial_v v_{h3} = 0$  on  $\Gamma_o$ , it suffices to equal to zero the following degrees of freedom :

$$\left. \begin{array}{l} \vec{v}_h(\Sigma) = \vec{0} \\ v_{h3,v}(\Sigma) = 0 \end{array} \right\} \forall \Sigma \text{ vertex of } \mathcal{T}_h \text{ located on } \Gamma_o . \quad (4.2.28)$$

Using the relations (4.2.21) and (4.2.22), the equations (4.2.28) involve

$$\left. \begin{aligned} \vec{A}(\Sigma) &= \vec{0} \\ B^\lambda(\Sigma) &= 0 \end{aligned} \right\} \begin{aligned} &\Psi\Sigma \text{ vertex of } \mathcal{C}_h \text{ located on } \Gamma_o . \end{aligned} \quad \begin{aligned} (4.2.29) \\ (4.2.30) \end{aligned}$$

The relation (4.2.29) combined with (4.2.23) involve

$$\vec{v}_h(\Sigma) = \vec{0} , \Psi\Sigma \text{ vertex of } \mathcal{C}_h \text{ located on } \Gamma_o . \quad (4.2.31)$$

Particularly, these relations (4.2.31) and the definition of the space  $\tilde{X}_{h1}$  involve

$$\tilde{v}_{h\alpha} = 0 \text{ on } \Gamma_o , \quad \alpha = 1, 2. \quad (4.2.32)$$

The relation (4.2.30) combined with (4.2.25) implies  $\vec{\omega}_h(\Sigma^+) \cdot \vec{a}^\lambda(\Sigma) = 0$ , and hence

$$\vec{\omega}_h(\Sigma^+) = C^{+\rightarrow 3}(\Sigma) ,$$

where  $C^+$  denotes a real number. Let us consider the components of the rotation vector  $\vec{\omega}_h(\Sigma^+)$  on the basis  $\vec{a}_{h1}^+$  ; we find

$$\tilde{\omega}_h^i(\Sigma^+) = C^{+\rightarrow 3}(\Sigma) \cdot \vec{a}_h^{i+} ,$$

or, by using (4.2.10)

$$\frac{1}{\sqrt{a_h^+}} e^{\lambda\beta} \tilde{v}_{h3,\beta}(\Sigma^+) = C^{+\rightarrow 3}(\Sigma) \cdot \vec{a}_h^{\lambda+} , \quad (4.2.33)$$

$$\frac{1}{2\sqrt{a_h^+}} e^{\lambda\beta} \tilde{v}_{h\beta,\lambda}(\Sigma^+) = C^{+\rightarrow 3}(\Sigma) \cdot \vec{a}_h^{3+} . \quad (4.2.34)$$

The hypothesis 4.2.1, i.e.,  $\vec{a}^3(\Sigma) \cdot \vec{a}_h^{3+} \neq 0 \quad \forall \Sigma$  vertex of  $\mathcal{T}_h$ , and the equation (4.2.34) determine the value of the constant  $C^+$  on the triangle  $K^+$  :

$$C^+ = \frac{e^{\lambda\beta}}{2\sqrt{a_h^+}} \frac{\tilde{v}_{h\beta,\lambda}(\Sigma^+)}{\vec{a}^3(\Sigma) \cdot \vec{a}_h^{3+}} , \quad (4.2.35)$$

where, following a previous remark,  $\tilde{v}_{h\beta,\lambda}(\Sigma^+)$  is only depending on the values of the functions  $\tilde{v}_{h\beta}$  at the vertices of triangle  $K^+$ . Then, the relations (4.2.33) (4.2.35) determine  $\tilde{v}_{h3,\beta}(\Sigma^+)$ , i.e.,

$$\tilde{v}_{h3,\beta}(\Sigma^+) = \frac{1}{2} e_{\lambda\beta} e^{\nu\mu} \frac{\vec{a}^3(\Sigma) \cdot \vec{a}_h^{\lambda+}}{\vec{a}^3(\Sigma) \cdot \vec{a}_h^{3+}} \tilde{v}_{h\mu,\nu}(\Sigma^+) , \quad \forall \Sigma \text{ vertex of } \mathcal{T}_h , \quad \left. \begin{array}{l} \\ \text{located on } \Gamma_o \end{array} \right\} (4.2.36)$$

This last relation prove that in general  $\tilde{v}_{h3,\beta}(\Sigma^+) \neq 0$  and hence

$$\tilde{v}_{h3}|_{\Gamma_o} \neq 0 \text{ and } \partial_\nu \tilde{v}_{h3}|_{\Gamma_o} \neq 0 . \quad (4.2.37)$$

Thus, the bijection  $F_h^{-1} : \vec{X}_h \rightarrow \vec{\tilde{X}}_h$  associates to the subspace  $\vec{V}_h$  of  $\vec{X}_h$  the following subspace  $\vec{\tilde{V}}_h$  of  $\vec{\tilde{X}}_h$  :

$$\vec{\tilde{V}}_h = \{ \vec{\tilde{v}}_h \in \vec{\tilde{X}}_h ; \vec{\tilde{v}}_h \text{ satisfies the relations (4.2.31)(4.2.36)} \} . \quad (4.2.38)$$

Particularly, this result means that the bijection  $F_h^{-1}$  associates to boundary conditions of *clamped type* for the functions of the space  $\vec{V}_h$ , a set of boundary conditions *different of clamped type* for the functions of the space  $\vec{\tilde{V}}_h$ .

Norms on the discrete spaces  $\vec{X}_h, \vec{V}_h, \vec{\tilde{X}}_h, \vec{\tilde{V}}_h$

The spaces  $\vec{X}_h, \vec{V}_h$  are respectively discrete subspaces of the spaces  $\vec{X} = (H^1(\Omega))^2 \times H^2(\Omega)$  and  $\vec{V}$  defined by (2.2.1). Hence, we equip these discrete spaces with the induced norm (2.2.2). Now, consider the application

$$\vec{v}_h \in \vec{X}_h \rightarrow \|\vec{v}_h\|_h = \left( \sum_{K \in \mathcal{T}_h} \left( \sum_{\alpha=1}^2 \|\tilde{v}_{h\alpha}\|_{1,K}^2 + \|\tilde{v}_{h3}\|_{2,K}^2 \right) \right)^{1/2} \quad (4.2.39)$$

which is well defined since, on the H.C.T.-triangle  $K = \bigcup_{i=1}^3 K_i$ , we have the property (CIARLET [5, Theorem 2.1.2]) :

$$\tilde{v}_{h3}|_K \in \mathcal{C}^1(K) \text{ and } \tilde{v}_{h3}|_{K_i} \in H^2(K_i) \Rightarrow \tilde{v}_{h3}|_K \in H^2(K) .$$

We can readily check that (4.2.39) defines a semi-norm on the space  $\vec{X}_h$ . Moreover,  $\|\vec{v}_h\|_h = 0$  implies  $\|\tilde{v}_{h\alpha}\|_{1,K} = 0$  and  $\|\tilde{v}_{h3}\|_{2,K} = 0$  so that  $\vec{v}_h = \vec{0}$ . Then the application (4.2.39) is a norm on the space  $\vec{X}_h$ . In the following we shall equip the subspace  $\vec{V}_h$  of  $\vec{X}_h$  with the corresponding induced norm.

#### 4.3 - Definition of the discrete problem using flat plate elements

Let us consider a displacement field  $\vec{v}_h \in \vec{V}_h$  of the faceted surface  $\mathcal{S}_h = \vec{\phi}_h(\Omega)$ . Then, like in (4.2.11) (4.2.12), we can define *triangle by triangle* the components of the corresponding strain tensor  $(\tilde{\gamma}_{h\alpha\beta})$  and the components of the change of curvature tensor  $(\tilde{\rho}_{h\alpha\beta})$ , i.e.,

$$\tilde{\gamma}_{h\alpha\beta}(\vec{v}_h) = \frac{1}{2} (\tilde{v}_{h\beta,\alpha} + \tilde{v}_{h\alpha,\beta}) \quad (4.3.1)$$

$$\tilde{\rho}_{h\alpha\beta}(\vec{v}_h) = \tilde{v}_{h3,\alpha\beta} \quad (4.3.2)$$

According to the expression (2.2.3) of the bilinear form  $a(.,.)$  a natural candidate for the *approximate bilinear form* is given by piecing together all the elementary energies, i.e.,

$$\begin{aligned} \tilde{a}_h(\vec{u}_h, \vec{v}_h) = & \sum_{K \in \mathcal{T}_h} \int_K \frac{Ee}{1-\nu^2} \{ (1-\nu) \tilde{\gamma}_{h\beta}^\alpha(\vec{u}_h) \tilde{\gamma}_{h\alpha}^\beta(\vec{v}_h) + \nu \tilde{\gamma}_{h\alpha}^\alpha(\vec{u}_h) \tilde{\gamma}_{h\beta}^\beta(\vec{v}_h) + \\ & + \frac{e^2}{12} [(1-\nu) \tilde{\rho}_{h\beta}^\alpha(\vec{u}_h) \tilde{\rho}_{h\alpha}^\beta(\vec{v}_h) + \nu \tilde{\rho}_{h\alpha}^\alpha(\vec{u}_h) \tilde{\rho}_{h\beta}^\beta(\vec{v}_h)] \} \sqrt{a_h} d\xi^1 d\xi^2, \quad \vec{u}_h, \vec{v}_h \in \vec{V}_h \end{aligned} \quad (4.3.3)$$

On this expression one can check that the approximate strain energy of the shell is *the sum of uncoupled elementary energies of plates* in extension, on the one hand, and in bending, on the other hand.

By analogy, we define the approximate linear form  $\tilde{f}_h$  according to the expression (2.2.4) of the linear form  $f(\cdot)$  :

$$\tilde{f}_h(\vec{v}_h) = \sum_{K \in \mathcal{T}_h} \int_K \vec{p} \cdot \vec{v}_h \sqrt{a_h} d\xi^1 d\xi^2, \quad \forall \vec{v}_h \in \vec{V}_h. \quad (4.3.4)$$

The integrals over the triangle  $K$  which appear in (4.3.3) and (4.3.4) are seldom computed exactly in practice, but rather *numerical integration schemes* are used. Thus, let us consider a numerical integration scheme over the reference triangle  $\hat{K}$  :

$$\int_{\hat{K}} \hat{\phi}(\hat{x}) d\hat{x} \sim \sum_{\ell=1}^L \hat{\omega}_\ell \hat{\phi}(\hat{b}_\ell). \quad (4.3.5)$$

All the integrals appearing in (4.3.3) and (4.3.4) are of the form  $\int_K \phi(x) dx$ .

We use the usual correspondence between  $\phi$  and  $\hat{\phi}$  through the affine invertible mapping

$$F_K : \hat{x} \in \hat{K} \rightarrow F_K(\hat{x}) = B_K \hat{x} + b_K \in K, \quad (4.3.6)$$

where  $B_K$  is an invertible matrix and  $b_K$  is a vector of  $\mathbb{R}^2$  such that

$$F_K(\hat{a}_i) = a_i, \quad i = 1, 2, 3, \quad (4.3.7)$$

where the points  $\hat{a}_i, a_i$  are the respective vertices of the triangles  $\hat{K}$  and  $K$ . Assuming, without loss of generality, that  $\det(B_K) > 0$ , we can write

$$\int_K \phi(x) dx = \det(B_K) \int_{\hat{K}} \hat{\phi}(\hat{x}) d\hat{x}. \quad (4.3.8)$$



Then, the numerical quadrature scheme over the triangle  $\hat{K}$  automatically induces a numerical quadrature scheme over the triangle  $K$ , namely,

$$\int_K \phi(x) dx \sim \sum_{\ell=1}^L \omega_{\ell,K} \phi(b_{\ell,K}) \quad (4.3.9)$$

with

$$\omega_{\ell,K} = \det(B_K) \hat{\omega}_{\ell} \quad \text{and} \quad b_{\ell,K} = F_K(\hat{b}_{\ell}), \quad 1 \leq \ell \leq L. \quad (4.3.10)$$

Moreover, we define the error functionals

$$\hat{E}(\hat{\phi}) = \int_{\hat{K}} \hat{\phi}(\hat{x}) d\hat{x} - \sum_{\ell=1}^L \hat{\omega}_{\ell} \hat{\phi}(\hat{b}_{\ell}), \quad (4.3.11)$$

$$E_K(\phi) = \int_K \phi(x) dx - \sum_{\ell=1}^L \omega_{\ell,K} \phi(b_{\ell,K}), \quad (4.3.12)$$

so that

$$E_K(\phi) = \det(B_K) \hat{E}(\hat{\phi}). \quad (4.3.13)$$

Thus, from the expression (4.3.3) we define a new approximate bilinear form :

$$\begin{aligned} a_h^*(\vec{u}_h, \vec{v}_h) = & \sum_{k \in \mathcal{T}_h} \sqrt{a_h} \left[ \sum_{\ell=1}^{L_1} \omega_{\ell,K} \left( \frac{Ee}{1-\nu^2} \{ (1-\nu) \tilde{\gamma}_{h\beta}^{\alpha}(\vec{u}_h) \tilde{\gamma}_{h\alpha}^{\beta}(\vec{v}_h) + \right. \right. \\ & \left. \left. + \nu \tilde{\gamma}_{h\alpha}^{\alpha}(\vec{u}_h) \tilde{\gamma}_{h\beta}^{\beta}(\vec{v}_h) \} \right) (b_{\ell_1,K}) \right. \\ & + \sum_{i=1}^3 \sum_{\ell_2=1}^{L_2} \omega_{\ell_2,K_i} \left( \frac{Ee^3}{12(1-\nu^2)} \{ (1-\nu) \tilde{\rho}_{h\beta}^{\alpha}(\vec{u}_h) \tilde{\rho}_{h\alpha}^{\beta}(\vec{v}_h) + \right. \\ & \left. \left. + \nu \tilde{\rho}_{h\alpha}^{\alpha}(\vec{u}_h) \tilde{\rho}_{h\beta}^{\beta}(\vec{v}_h) \} \right) (b_{\ell_2,K_i}) \right]. \end{aligned} \quad (4.3.14)$$

In the last expression we have used *two different numerical quadrature schemes* to take into account the *uncoupling* between the membrane terms, i.e.,  $\tilde{\gamma}_{h\alpha}^{\beta}$ , and the bending terms, i.e.,  $\tilde{\rho}_{h\alpha}^{\beta}$ , each of them being approximated in two different finite element spaces.

Practically, we can use effectively two different numerical schemes or we can use only one numerical scheme satisfying the conditions that we shall derive later for the membrane and bending parts. Moreover, the second part of the second member of the expression (4.3.14) takes into account the definition of the function  $\tilde{v}_{h3}$  subtriangle  $K_i$  by subtriangle  $K_i$  - see (4.2.4) -.

By analogy, we define a new approximate linear form

$$f_h^*(\vec{v}_h) = \sum_{K \in \mathcal{T}_h} \left[ \sum_{\ell_1=1}^{L_1} \omega_{\ell_1, K} (\{\vec{p}, \vec{a}_h^\alpha\} \tilde{v}_{h\alpha} \sqrt{a_h}) (b_{\ell_1, K}) + \sum_{i=1}^3 \sum_{\ell_2=1}^{L_2} \omega_{\ell_2, K_i} (\{\vec{p}, \vec{a}_h^3\} \tilde{v}_{h3} \sqrt{a_h}) (b_{\ell_2, K_i}) \right] \quad (4.3.15)$$

Then, using the bijection  $F_h : \vec{v}_h \in \vec{V}_h \rightarrow \vec{v}_h \in \vec{V}_h$  defined in the theorem 4.2.1, we can define the following approximate forms :

$$b_h(\vec{u}_h, \vec{v}_h) = a_h^*(\vec{u}_h, \vec{v}_h) \quad (4.3.16)$$

$$g_h(\vec{v}_h) = f_h^*(\vec{v}_h) \quad (4.3.17)$$

for all  $\vec{u}_h, \vec{v}_h \in \vec{V}_h$  and all  $\vec{u}_h, \vec{v}_h \in \vec{V}_h$  associated through the bijection  $F_h$ .

Now the discrete problem using flat plate elements can be stated as follows : Find an element  $\vec{u}_h^* \in \vec{V}_h$  such that

$$a_h^*(\vec{u}_h^*, \vec{v}_h) = f_h^*(\vec{v}_h) \quad , \quad \forall \vec{v}_h \in \vec{V}_h \quad , \quad (4.3.18)$$

or, equivalently, using the bijection  $F_h : \vec{V}_h \rightarrow \vec{V}_h$  and the definitions (4.3.16) (4.3.17) : Find an element  $\vec{u}_h^* \in \vec{V}_h$  such that

$$b_h(\vec{u}_h^*, \vec{v}_h) = g_h(\vec{v}_h) \quad , \quad \forall \vec{v}_h \in \vec{V}_h \quad . \quad (4.3.19)$$

#### 4.4 - Another alternative : CARR's method

The ideas are similar to those of Section 4.2. Thus, we shall emphasize on the new aspects of the definition of the space  $\tilde{V}_h$ .

Definition of the space  $\tilde{X}_h$  :

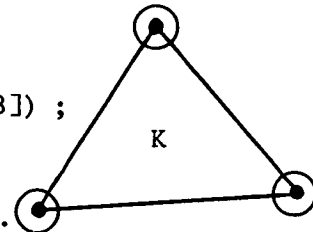
According to relation (4.2.1) we have to define the discrete spaces  $\tilde{X}_{h1}$  and  $\tilde{X}_{h2}$ .

Space  $\tilde{X}_{h1}$  : this space is defined following the lines of (4.2.3) but now

(i)  $v_h|_K \in P_3^1(K)$  (CIARLET [5, page 68]) ;

(ii)  $v_h|_K$  is completely specified by

the values  $v_h|_K(\Sigma_i)$ ,  $Dv_h|_K(\Sigma_i)$ ,  $i = 1, 2, 3$ .



(4.4.1)

□

Moreover, the space  $\tilde{X}_{h2}$  is the one defined in (4.2.4). Denoting by  $M_h$  the number of vertices of the triangulation, the space  $\tilde{X}_h$  has now  $9M_h$  degrees of freedom.

#### Derivation of the compatibility conditions :

Let us consider again two facets  $k^+$  and  $k^-$  of  $\mathcal{S}_h$  which have a common vertex  $\sigma$  (see Figure 4.1.1). According as we regard this common vertex as a point of the middle surface  $\mathcal{S}$  or as a vertex of the facet  $k^+$ , the displacement field and the associated rotation vector, strain tensor and change of curvature tensor are respectively given by the relations (4.2.5) to (4.2.8) on the one hand and by the relations (4.2.9) to (4.2.12) on the other hand. Moreover for the triangle  $K^-$  we have similar results to those given by relations (4.2.9) to (4.2.12) ; it suffices to substitute the superscript - to the superscript + .

One can easily check that to any function  $\vec{v}_h \in \vec{V}_h$ , we can associate the set of values of  $\vec{v}_h$ ,  $\vec{\omega}(\vec{v}_h)$ ,  $\gamma_{\alpha\beta}(\vec{v}_h)$  at all the vertices  $\Sigma$  of the triangulation  $\mathcal{T}_h$  and, conversely, to the set of values of  $\vec{v}_h$ ,  $\vec{\omega}(\vec{v}_h)$ ,  $\gamma_{\alpha\beta}(\vec{v}_h)$  at all the vertices  $\Sigma$  of the triangulation  $\mathcal{T}_h$  we can associate one and only one function  $\vec{v}_h \in \vec{V}_h$ . In the same way, we can show that the knowledge of any function  $\vec{v}_h \in \vec{V}_h$  is equivalent to the knowledge of  $\vec{v}_h$ ,  $\vec{\omega}_h(\vec{v}_h)$ ,  $\gamma_{h\alpha\beta}(\vec{v}_h)$  at any vertex  $\Sigma$  of any triangle  $K \in \mathcal{T}_h$ . Thus, we derive the *compatibility conditions* by writting that :

(i) the displacement  $\vec{v}_h$  is continuous at the vertices  $\Sigma$  of the triangulation  $\mathcal{T}_h$ , i.e.,

$$\vec{v}_h(\Sigma^+) = \vec{v}_h(\Sigma^-) , \quad \forall \Sigma \text{ vertex of } \mathcal{T}_h . \quad (4.4.2)$$

(ii) the rotation vectors  $\vec{\omega}_h(\vec{v}_h)$  are continuous at the vertices  $\Sigma$  of the triangulation  $\mathcal{T}_h$ , i.e.,

$$\vec{\omega}_h(\Sigma^+) = \vec{\omega}_h(\Sigma^-) , \quad \forall \Sigma \text{ vertex of } \mathcal{T}_h . \quad (4.4.3)$$

(iii) The tangential components (with respect to the middle surface  $\mathcal{S}$ ) of the strain tensors  $(\gamma_{hij}^*(\vec{v}_h))$  are continuous at the vertices  $\Sigma$  of the triangulation  $\mathcal{T}_h$ , i.e.,

$$\left. \begin{aligned} \gamma_{hij}^*(\Sigma^+) (\vec{a}_h^{i+} \cdot \vec{a}_\alpha(\Sigma)) (\vec{a}_h^{j+} \cdot \vec{a}_\beta(\Sigma)) \vec{a}^\alpha(\Sigma) \otimes \vec{a}^\beta(\Sigma) = \\ = \gamma_{hij}^*(\Sigma^-) (\vec{a}_h^{i-} \cdot \vec{a}_\alpha(\Sigma)) (\vec{a}_h^{j-} \cdot \vec{a}_\beta(\Sigma)) \vec{a}^\alpha(\Sigma) \otimes \vec{a}^\beta(\Sigma), \end{aligned} \right\} \quad (4.4.4)$$

$\forall \Sigma \text{ vertex of } \mathcal{T}_h , \alpha \text{ and } \beta \text{ given (no summation)}$

This expression uses all the components of the strain tensors  $\gamma_{hij}^*$  on the basis  $\vec{a}_h^i$  and so we are led to give some complements on the definition of this tensor. For details we refer the reader to KOITER [14] or BERNADOU-CIARLET [15]. Let us consider the shell  $\mathcal{C}$  as a tridimensional continuous medium parameterized by a system of curvilinear coordinates  $(\xi^1, \xi^2, \xi^3)$ , i.e.,

$$\mathcal{C} = \{M \in \mathcal{C}^3 ; \vec{OM} = \vec{\phi}(\xi^1, \xi^2) + \xi^3 \vec{a}_3, (\xi^1, \xi^2) \in \Omega, \quad$$

$$- \frac{1}{2} e(\xi^1, \xi^2) \leq \xi^3 \leq \frac{1}{2} e(\xi^1, \xi^2)\} ,$$

where  $e$  denotes the thickness of the shell. Particularly, the middle surface  $\mathcal{S}$  of the shell is obtained for  $\xi^3 = 0$ .

Then, let us assume that the shell becomes deformed. The strain tensor  $(\gamma^*) = \gamma_{ij}^* \vec{g}^i \otimes \vec{g}^j$  is given by

$$\gamma_{ij}^* = \frac{1}{2} (\bar{g}_{ij} - g_{ij}) \quad (4.4.5)$$

where  $(\vec{g}^i)$  denotes the contravariant basis of the three-dimensional medium,  $\bar{g}_{ij}$  (resp.  $g_{ij}$ ) the metric tensor of the continuous medium in deformed (resp. undeformed) position, for a *same* parameterization  $(\xi^1, \xi^2, \xi^3)$  of the two configurations of the medium. The exponent  $*$  which appears in (4.4.5) distinguishes the tensors defined on  $\mathcal{C}^3$  from the tensors defined on the middle surface. According to KOITER [14] we assume that

(i) the normals to the undeformed middle surface, considered as point sets of the shell, remain normal to the deformed middle surface ;

(ii) during the deformation the stresses are approximatively plane and parallel to the tangent plane to the middle surface.

More precisely, if we denote by  $M$  the position of any particle of  $\mathcal{C}$ , by  $P$  the projection of  $M$  on the middle surface  $\mathcal{S}$  of  $\mathcal{C}$ , by  $\bar{M}$  and  $\bar{P}$  the respective positions of the same particles after deformation, then the first assumption (i) involves

$$\vec{OM} = \vec{OP} + \xi^3 \vec{a}_3$$

where the three vectors

$$\vec{a}_\alpha = \vec{OP}_{,\alpha}, \quad \vec{a}_3 = \frac{\vec{a}_1 \times \vec{a}_2}{|\vec{a}_1 \times \vec{a}_2|}$$

constitute a local basis of  $\mathcal{C}^3$  analogous to the basis  $(\vec{a}_i)$  introduced in paragraph 2, and where  $\bar{\xi}^3$  is the coordinate of  $\bar{M}$  measured on  $(\bar{P}, \vec{a}_3)$ .

In first approximation, we can assume that  $\bar{\xi}^3$  is independent from  $\xi^1, \xi^2$  and just dependent on  $\xi^3$ .

Then, one can show that

$$g_{\alpha\beta} = \vec{g}_\alpha \cdot \vec{g}_\beta = a_{\alpha\beta} - 2b_{\alpha\beta}\xi^3 + c_{\alpha\beta}(\xi^3)^2 ,$$

$$\bar{g}_{\alpha\beta} = \vec{\bar{g}}_\alpha \cdot \vec{\bar{g}}_\beta = \bar{a}_{\alpha\beta} - 2\bar{b}_{\alpha\beta}\bar{\xi}^3 + \bar{c}_{\alpha\beta}(\bar{\xi}^3)^2 ,$$

$$g_{\alpha 3} = g_{3\alpha} = \bar{g}_{\alpha 3} = \bar{g}_{3\alpha} = 0 ,$$

$$g_{33} = 1 , \quad \bar{g}_{33} = \left( \frac{d\bar{\xi}^3}{d\xi^3} \right)^2 .$$

Using these relations and the definition (4.4.5), the strain tensor  $(\gamma^*)$  has the following components on the middle surface  $\mathcal{L} (\xi^3 = \bar{\xi}^3 = 0)$  :

$$\left. \begin{aligned} \gamma_{\alpha\beta}^* &= \frac{1}{2} (\bar{a}_{\alpha\beta} - a_{\alpha\beta}) = \gamma_{\alpha\beta} \\ \gamma_{\alpha 3}^* &= \gamma_{3\alpha}^* = 0 \\ \gamma_{33}^* &= \frac{1}{2} \left[ \left( \frac{d\bar{\xi}^3}{d\xi^3} (0) \right)^2 - 1 \right] . \end{aligned} \right\}$$

Moreover, for an elastic isotropic homogeneous medium verifying the Hooke's law, one can prove that the plane stress hypothesis (ii) involves

$$\gamma_{33}^* = - \frac{\nu}{1-\nu} \gamma_\alpha^{\alpha*} \quad \text{with} \quad \gamma_\alpha^{\alpha*} = g^{\alpha\lambda} \gamma_{\alpha\lambda}^* ,$$

where  $\nu$  is the POISSON coefficient of the medium. Therefore on the middle surface  $\mathcal{L}$  we have

$$\gamma_{33}^* = - \frac{\nu}{1-\nu} a^{\alpha\lambda} \gamma_{\alpha\lambda} .$$

To summarize, the strain tensor  $(\gamma^*) = \gamma_{ij}^* \vec{g}^i \otimes \vec{g}^j$  of the three-dimensional medium  $\mathcal{C}$  has the following expression on the middle surface  $\mathcal{S}$  of the shell :

$$\begin{aligned} (\gamma^*) &= \gamma_{ij}^* \vec{a}^i \otimes \vec{a}^j , \\ \gamma_{\alpha\beta}^* &= \gamma_{\alpha\beta} ; \gamma_{\alpha 3}^* = \gamma_{3\alpha}^* = 0 ; \gamma_{33}^* = - \frac{\nu}{1-\nu} a^{\alpha\lambda} \gamma_{\alpha\lambda} . \end{aligned} \quad \left. \vphantom{\begin{aligned} (\gamma^*) &= \gamma_{ij}^* \vec{a}^i \otimes \vec{a}^j , \\ \gamma_{\alpha\beta}^* &= \gamma_{\alpha\beta} ; \gamma_{\alpha 3}^* = \gamma_{3\alpha}^* = 0 ; \gamma_{33}^* = - \frac{\nu}{1-\nu} a^{\alpha\lambda} \gamma_{\alpha\lambda} . \end{aligned}} \right\} (4.4.6)$$

Naturally, all the previous complements can be applied to the particular cases for which the middle surface of the shell is either the facet  $k^+$  or the facet  $k^-$ . Particularly, the relations (4.4.6) show that the knowledge of the strain surface tensor  $\gamma_{\alpha\beta}$  involves the knowledge of the  $\mathcal{C}^3$ -tensor  $\gamma_{ij}^*$  upon the middle surface, and then, the relation (4.4.4) takes a sense.

Let  $\tilde{X}_h$  be the following space

$$\tilde{X}_h = \{ \tilde{v}_h \in \tilde{X}_h ; \tilde{v}_h \text{ satisfies the compatibility conditions } (4.4.2) \text{ to } (4.4.4) \} . \quad (4.4.7)$$

Let us state the following hypothesis :

Hypothesis 4.4.1 : Let  $\sigma = \sigma^+ = \sigma^-$  be a vertex of the faceted surface  $\mathcal{I}_h$  respectively considered as a point of  $\mathcal{S}$ , as a vertex of a facet  $k^+$  and as a vertex of a facet  $k^-$ . By using the correspondence  $\sigma = \vec{\phi}(\Sigma)$ , we denote  $\vec{a}^3(\Sigma)$ ,  $\vec{a}_h^{3+}$ ,  $\vec{a}_h^{3-}$  the corresponding normal vector to the middle surface  $\mathcal{S}$ , to the facet  $k^+$  and to the facet  $k^-$ . Then, if we set for any vertex  $\Sigma$  of two adjacent triangles  $K^+$  and  $K^-$

$$\cos \delta_3^+ = \vec{a}^3(\Sigma) \cdot \vec{a}_h^{3+} , \quad \cos \delta_3^- = \vec{a}^3(\Sigma) \cdot \vec{a}_h^{3-} ,$$

we assume that

$$\cos \delta_3^+ > 0 , \quad \cos \delta_3^- > 0 \quad \text{if } -1 < \nu \leq 0$$

$$\cos \delta_3^+ > \sqrt{\nu} , \quad \cos \delta_3^- > \sqrt{\nu} \quad \text{if } 0 < \nu < \frac{1}{2}$$

where  $\nu$  is the POISSON coefficient of the medium.

□

Theorem 4.4.1 : We assume that the hypothesis 4.4.1 is satisfied.  
Then, there exists a "suitable" bijection  $F_h$  between the space  $\tilde{X}_h$ ,  
defined by (4.4.7), and the space  $\vec{X}_h = X_{h1} \times X_{h1} \times X_{h2}$ , where  $X_{h1}$   
denotes the conforming finite element space ( $X_{h1} \subset C^0(\bar{\Omega})$ ) associated  
to the  $P'_3$ -Hermite triangle and  $X_{h2}$  is defined by (3.1.5).

Proof : We indicate only the main points of the proof which is similar to the one of theorem 4.2.1.

(i)  $\vec{v}_h \in \tilde{X}_h \rightarrow \vec{v}_h \in \vec{X}_h :$

Let us consider any function  $\vec{v}_h \in \tilde{X}_h$ , i.e.,  $\vec{v}_h \in \tilde{X}_h$  and  $\vec{v}_h$  satisfies the relations (4.4.2) to (4.4.4). To the set of values of the degrees of freedom of the function  $\vec{v}_h$  we associate the following set of values

$$\vec{A}(\Sigma) = \vec{v}_h(\Sigma^+) = \vec{v}_h(\Sigma^-) , \quad \forall \Sigma \text{ vertex of } \mathcal{T}_h , \quad (4.4.8)$$

$$\vec{B}(\Sigma) = \vec{\omega}_h(\Sigma^+) = \vec{\omega}_h(\Sigma^-) , \quad \forall \Sigma \text{ vertex of } \mathcal{T}_h , \quad (4.4.9)$$

$$\left. \begin{aligned} C_{\alpha\beta}(\Sigma) &= (\vec{a}_h^{i+} \cdot \vec{a}_\alpha(\Sigma)) (\vec{a}_h^{j+} \cdot \vec{a}_\beta(\Sigma)) \tilde{\gamma}_{hij}^*(\Sigma^+) = \\ &= (\vec{a}_h^{i-} \cdot \vec{a}_\alpha(\Sigma)) (\vec{a}_h^{j-} \cdot \vec{a}_\beta(\Sigma)) \tilde{\gamma}_{hij}^*(\Sigma^-) \end{aligned} \right\} \quad (4.4.10)$$

In the last expression we have, following relations (4.2.11) and (4.4.6) :

$$\left. \begin{aligned} \tilde{\gamma}_{h\alpha\beta}^*(\Sigma^+) &= \tilde{\gamma}_{h\alpha\beta}(\Sigma^+) = \frac{1}{2} (\tilde{v}_{h\beta,\alpha}^+ + \tilde{v}_{h\alpha,\beta}^+) \\ \tilde{\gamma}_{h\alpha 3}^*(\Sigma^+) &= \tilde{\gamma}_{h3\alpha}^*(\Sigma^+) = 0 \\ \tilde{\gamma}_{h33}^*(\Sigma^+) &= - \frac{\nu}{1-\nu} a_h^{\alpha\lambda+} \tilde{\gamma}_{h\alpha\lambda}(\Sigma^+) \end{aligned} \right\} \quad (4.4.11)$$

and similar expressions at point  $\Sigma^-$ .



Then, to the set of values (4.4.8) to (4.4.10), we associate the function  $\vec{v}_h \in \vec{X}_h$  determined by the following values of its degrees of freedom at any vertex  $\Sigma$  of  $\mathcal{T}_h$  :

$$\vec{v}_h(\Sigma) = \vec{A}(\Sigma) , \quad (4.4.12)$$

$$v_{h3,v}(\Sigma) = \varepsilon_{\alpha v}(\Sigma) B^\alpha(\Sigma) - b_v^\alpha(\Sigma) \vec{A}(\Sigma) \cdot \vec{a}_\alpha(\Sigma) , \quad (4.4.13)$$

$$v_{h1,1}(\Sigma) = c_{11}(\Sigma) + b_{11}(\Sigma) v_{h3}(\Sigma) + \Gamma_{11}^\lambda(\Sigma) v_{h\lambda}(\Sigma) , \quad (4.4.14)$$

$$v_{h2,2}(\Sigma) = c_{22}(\Sigma) + b_{22}(\Sigma) v_{h3}(\Sigma) + \Gamma_{22}^\lambda(\Sigma) v_{h\lambda}(\Sigma) , \quad (4.4.15)$$

$$v_{h1,2}(\Sigma) = c_{12}(\Sigma) + b_{12}(\Sigma) v_{h3}(\Sigma) + \Gamma_{12}^\lambda(\Sigma) v_{h\lambda}(\Sigma) - \sqrt{a(\Sigma)} \vec{B}(\Sigma) \cdot \vec{a}^3(\Sigma) , \quad (4.4.16)$$

$$v_{h2,1}(\Sigma) = c_{12}(\Sigma) + b_{12}(\Sigma) v_{h3}(\Sigma) + \Gamma_{12}^\lambda(\Sigma) v_{h\lambda}(\Sigma) + \sqrt{a(\Sigma)} \vec{B}(\Sigma) \cdot \vec{a}^3(\Sigma) . \quad (4.4.17)$$

To derive the previous relations, we use the relations (4.2.19) and (4.2.20) to get (4.4.12) and (4.4.13), we use the relations (2.1.13) and (4.2.7) to get (4.4.14) and (4.4.15), and finally, we use the relations (2.1.13)(4.2.6) and (4.2.7) to get the two last, i.e., (4.4.16) (4.4.17).

Thus, to any given function  $\vec{v}_h \in \vec{X}_h$ , the relations (4.4.12) to (4.4.17) associate one and only one function  $\vec{v}_h \in \vec{X}_h$ . Let us consider now the converse property :

$$(ii) \vec{v}_h \in \vec{X}_h \rightarrow \vec{v}_h \in \vec{X}_h :$$

The knowledge of  $\vec{v}_h \in \vec{X}_h$  is equivalent to the knowledge of its degrees of freedom  $\vec{v}_h(\Sigma)$  and  $v_{hi,\alpha}(\Sigma)$  for all vertices  $\Sigma$  of  $\mathcal{T}_h$ . Then, the first members of the equations (4.4.12) to (4.4.17) are known and we can easily derive the following parameters as unique solution of the system (4.4.12) to (4.4.17) :

$$\vec{A}(\Sigma) = \vec{v}_h(\Sigma) \quad (4.4.18)$$

$$B^\lambda(\Sigma) = \varepsilon^{\lambda\beta}(\Sigma) [v_{h3,\beta}(\Sigma) + b_\beta^\alpha(\Sigma)v_{h\alpha}(\Sigma)] , \quad (4.4.19)$$

$$B^3(\Sigma) = \frac{1}{2\sqrt{a}(\Sigma)} [v_{h2,1}(\Sigma) - v_{h1,2}(\Sigma)] \quad \left. \vphantom{B^3(\Sigma)} \right\} \text{i.e., } \vec{B}(\Sigma) , \quad (4.4.20)$$

$$C_{\alpha\beta}(\Sigma) = \frac{1}{2} (v_{h\alpha,\beta}(\Sigma) + v_{h\beta,\alpha}(\Sigma)) - \Gamma_{\alpha\beta}^\lambda(\Sigma)v_{h\lambda}(\Sigma) - b_{\alpha\beta}(\Sigma)v_{h3}(\Sigma) . \quad (4.4.21)$$

From relation (4.4.18) we derive  $\vec{v}_h(\Sigma^+)$  and  $\vec{v}_h(\Sigma^-)$  as solutions of the equations

$$\vec{v}_h(\Sigma^+) = \vec{v}_h(\Sigma^-) = \vec{A}(\Sigma) . \quad (4.4.22)$$

Next, the equations

$$\vec{\omega}_h(\Sigma^+) = \vec{\omega}_h(\Sigma^-) = \vec{B}(\Sigma) \quad (4.4.23)$$

permit to find

$$\vec{v}_{h3,\beta}^+(\Sigma) = e_{\lambda\beta} \sqrt{a_h^+} \vec{B}(\Sigma) \cdot \vec{a}_h^{\lambda+} \quad (4.4.24)$$

$$\frac{1}{2} (\vec{v}_{h2,1}^+(\Sigma) - \vec{v}_{h1,2}^+(\Sigma)) = \sqrt{a_h^+} \vec{B}(\Sigma) \cdot \vec{a}_h^{3+} \quad (4.4.25)$$

and similar results for  $\vec{v}_{h3,\beta}^-$  and  $\frac{1}{2} (\vec{v}_{h2,1}^- - \vec{v}_{h1,2}^-)$ .

Finally, we prove in lemma (4.4.I) that the set of equations

$$\tilde{\gamma}_{h\alpha 3}^*(\Sigma^+) = \tilde{\gamma}_{h3\alpha}^*(\Sigma^+) = \tilde{\gamma}_{h\alpha 3}^*(\Sigma^-) = \tilde{\gamma}_{h3\alpha}^*(\Sigma^-) = 0 , \quad (4.4.26)$$

$$\left. \begin{aligned} \tilde{\gamma}_{h33}^*(\Sigma^+) &= -\frac{\nu}{1-\nu} a_h^{\alpha\lambda+}(\Sigma^+) \tilde{\gamma}_{h\alpha\lambda}^*(\Sigma^+) , \\ \tilde{\gamma}_{h33}^*(\Sigma^-) &= -\frac{\nu}{1-\nu} a_h^{\alpha\lambda-}(\Sigma^-) \tilde{\gamma}_{h\alpha\lambda}^*(\Sigma^-) , \end{aligned} \right\} (4.4.27)$$

$$\left. \begin{aligned} &(\vec{a}_h^{i+} \cdot \vec{a}_\alpha^+(\Sigma)) (\vec{a}_h^{j+} \cdot \vec{a}_\beta^+(\Sigma)) \tilde{\gamma}_{hij}^*(\Sigma^+) \\ &= (\vec{a}_h^{i-} \cdot \vec{a}_\alpha^-(\Sigma)) (\vec{a}_h^{j-} \cdot \vec{a}_\beta^-(\Sigma)) \tilde{\gamma}_{hij}^*(\Sigma^-) = C_{\alpha\beta}(\Sigma) \end{aligned} \right\} (4.4.28)$$

has unique solution  $\tilde{\gamma}_{hij}^*(\Sigma^+)$  and  $\tilde{\gamma}_{hij}^*(\Sigma^-)$ . We recall that equations (4.4.26) (4.4.27) originate from the fundamental hypothesis of KOITER's theory. Then the equations (4.4.11) and (4.4.25) imply :

$$\left. \begin{aligned} \tilde{v}_{h1,1}^+ &= \tilde{\gamma}_{h11}^*(\Sigma^+) , \quad \tilde{v}_{h2,2}^+ = \tilde{\gamma}_{h22}^*(\Sigma^+) \\ \tilde{v}_{h2,1}^+ &= \tilde{\gamma}_{h12}^*(\Sigma^+) + \sqrt{a_h^+} \vec{B}(\Sigma) \cdot \vec{a}_h^{3+} \\ \tilde{v}_{h1,2}^+ &= \tilde{\gamma}_{h12}^*(\Sigma^+) - \sqrt{a_h^+} \vec{B}(\Sigma) \cdot \vec{a}_h^{3+} \end{aligned} \right\} (4.4.29)$$

and similar results for  $\tilde{v}_{h\alpha,\beta}^-$ .

Thus the equations (4.4.22) (4.4.24) (4.4.29) allow to associate one and only one value to all the degrees of freedom of the space  $\tilde{X}_h$ . The corresponding function  $\tilde{v}_h$  belongs in fact to the subspace  $\tilde{X}_h$  of  $\tilde{X}_h$  since the function  $\tilde{v}_h$  satisfies by construction the compatibility conditions (4.4.2) to (4.4.4) (see equations (4.4.22), (4.4.23) and (4.4.28)).

□

Lemma 4.4.1 : Under the hypothesis 4.4.1, the set of equations (4.4.26) to (4.4.28) has unique solutions  $\tilde{\gamma}_{hij}^*(\Sigma^+)$  and  $\tilde{\gamma}_{hij}^*(\Sigma^-)$ .

Proof : The equations giving  $\tilde{\gamma}_{hij}^*(\Sigma^+)$  on the one hand and the equations giving  $\tilde{\gamma}_{hij}^*(\Sigma^-)$  on the other hand are not coupled. Therefore, we just consider the determination of  $\tilde{\gamma}_{hij}^*(\Sigma^+)$  as a unique solution of the equations

$$(\vec{a}_h^{i+} \cdot \vec{a}_\alpha(\Sigma)) (\vec{a}_h^{j+} \cdot \vec{a}_\beta(\Sigma)) \tilde{\gamma}_{hij}^*(\Sigma^+) = c_{\alpha\beta}(\Sigma) \quad (4.4.30)$$

$$\tilde{\gamma}_{h\alpha 3}^*(\Sigma^+) = \tilde{\gamma}_{h3\alpha}^*(\Sigma^+) = 0 \quad (4.4.31)$$

$$\tilde{\gamma}_{h33}^*(\Sigma^+) = - \frac{\nu}{1-\nu} a_h^{\lambda\mu+}(\Sigma^+) \tilde{\gamma}_{h\lambda\mu}^*(\Sigma^+) \quad (4.4.32)$$

Substituting equations (4.4.31) (4.4.32) into the equation (4.4.30) and taking into account the symmetry  $\tilde{\gamma}_{h\alpha\beta}^* = \tilde{\gamma}_{h\beta\alpha}^*$  we find the following system of three equations to three unknowns (for convenience, we have dropped the arrows and the references to  $\Sigma$  and  $\Sigma^+$ ) :

(4.4.33)

$$\begin{aligned}
 & [ (a_h^{1+} \cdot a_1)^2 - \frac{v}{1-v} a_h^{11+} (a_h^{3+} \cdot a_1)^2 ] \gamma_{h11}^* + \\
 & + 2 [ (a_h^{1+} \cdot a_1) (a_h^{2+} \cdot a_1) - \frac{v}{1-v} a_h^{12+} (a_h^{3+} \cdot a_1)^2 ] \gamma_{h12}^* + \\
 & + [ (a_h^{2+} \cdot a_1)^2 - \frac{v}{1-v} a_h^{22+} (a_h^{3+} \cdot a_1)^2 ] \gamma_{h22}^* = c_{11} \\
 & [ (a_h^{1+} \cdot a_1) (a_h^{1+} \cdot a_2) - \frac{v}{1-v} a_h^{11+} (a_h^{3+} \cdot a_1) (a_h^{3+} \cdot a_2) ] \gamma_{h11}^* + \\
 & + [ (a_h^{1+} \cdot a_1) (a_h^{2+} \cdot a_2) + (a_h^{2+} \cdot a_1) (a_h^{1+} \cdot a_2) - \frac{2v}{1-v} a_h^{12+} (a_h^{3+} \cdot a_1) (a_h^{3+} \cdot a_2) ] \gamma_{h12}^* + \\
 & + [ (a_h^{2+} \cdot a_1) (a_h^{2+} \cdot a_2) - \frac{v}{1-v} a_h^{22+} (a_h^{3+} \cdot a_1) (a_h^{3+} \cdot a_2) ] \gamma_{h22}^* = c_{12} \\
 & [ (a_h^{1+} \cdot a_2)^2 - \frac{v}{1-v} a_h^{11+} (a_h^{3+} \cdot a_2)^2 ] \gamma_{h11}^* + \\
 & + 2 [ (a_h^{1+} \cdot a_2) (a_h^{2+} \cdot a_2) - \frac{v}{1-v} a_h^{12+} (a_h^{3+} \cdot a_2)^2 ] \gamma_{h12}^* + \\
 & + [ (a_h^{2+} \cdot a_2)^2 - \frac{v}{1-v} a_h^{22+} (a_h^{3+} \cdot a_2)^2 ] \gamma_{h22}^* = c_{22} .
 \end{aligned}$$

To prove the lemma 4.4.1 it suffices to show that the determinant of the system (4.4.33) is different from 0. In this goal, we set with obvious notations :

$$\begin{aligned}
 a_h^{1+} \cdot a_1 &= \sqrt{a_h^{11+}} a_{11} \cos \alpha_1 , & a_h^{2+} \cdot a_1 &= \sqrt{a_h^{22+}} a_{11} \cos \beta_1 \\
 a_h^{1+} \cdot a_2 &= \sqrt{a_h^{11+}} a_{22} \cos \alpha_2 , & a_h^{2+} \cdot a_2 &= \sqrt{a_h^{22+}} a_{22} \cos \beta_2 \\
 a_h^{1+} \cdot a_3 &= \sqrt{a_h^{11+}} \cos \alpha_3 , & a_h^{2+} \cdot a_3 &= \sqrt{a_h^{22+}} \cos \beta_3
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} a_h^{1+} \cdot a_1 &= \sqrt{a_h^{11+}} a_{11} \cos \alpha_1 , \\ a_h^{1+} \cdot a_2 &= \sqrt{a_h^{11+}} a_{22} \cos \alpha_2 , \\ a_h^{1+} \cdot a_3 &= \sqrt{a_h^{11+}} \cos \alpha_3 , \end{aligned}} \right\} (4.4.34)$$

$$a_h^{3+} \cdot a_1 = \sqrt{a_{11}^{12+}} \cos \gamma_1, \quad a_h^{3+} \cdot a_2 = \sqrt{a_{22}^{12+}} \cos \gamma_2 \quad (4.4.35)$$

$$A = \frac{a_h^{12+}}{\sqrt{a_h^{11+} a_h^{22+}}}, \quad B = \frac{-v}{1-v} \quad (4.4.36)$$

Then the determinant D of the system (4.4.33) can be written as

$$D = (a_h^{11+} a_h^{22+} a_{11} a_{22})^{\frac{3}{2}} \begin{bmatrix} \cos^2 \alpha_1 + B \cos^2 \gamma_1 & 2[\cos \alpha_1 \cos \beta_1 + A B \cos^2 \gamma_1] \cos^2 \beta_1 + B \cos^2 \gamma_1 & \\ \cos \alpha_1 \cos \alpha_2 + B \cos \gamma_1 \cos \gamma_2 & \cos \alpha_1 \cos \beta_2 + \cos \alpha_2 \cos \beta_1 + 2 A B \cos \gamma_1 \cos \gamma_2 & \cos \beta_1 \cos \beta_2 + B \cos \gamma_1 \cos \gamma_2 \\ \cos^2 \alpha_2 + B \cos^2 \gamma_2 & 2[\cos \alpha_2 \cos \beta_2 + A B \cos^2 \gamma_2] \cos^2 \beta_2 + B \cos^2 \gamma_2 & \end{bmatrix}$$

After a technical calculation, we get

$$D = (a_h^{11+} a_h^{22+} a_{11} a_{22})^{\frac{3}{2}} \left\{ \begin{aligned} & (\cos \alpha_1 \cos \beta_2 - \cos \alpha_2 \cos \beta_1) \\ & \{ (\cos \alpha_1 \cos \beta_2 - \cos \alpha_2 \cos \beta_1)^2 + B(1-A^2) (\cos \alpha_2 \cos \gamma_1 - \cos \alpha_1 \cos \gamma_2)^2 \\ & + B[(\cos \beta_2 \cos \gamma_1 - \cos \beta_1 \cos \gamma_2) - A(\cos \alpha_2 \cos \gamma_1 - \cos \alpha_1 \cos \gamma_2)]^2 \} \end{aligned} \right\} \quad (4.4.37)$$

Now, we are going to transform D in order to get a very simple expression. Let us set

$$a_3 \cdot a_{h1}^+ = \sqrt{a_{h11}^+} \cos \delta_1, \quad a_3 \cdot a_{h2}^+ = \sqrt{a_{h22}^+} \cos \delta_2, \quad a_3 \cdot a_{h3}^+ = \cos \delta_3. \quad (4.4.38)$$

Moreover, we have

$$a_i = (a_i \cdot a_h^{j+}) a_{hj}^+.$$

Then

$$a_1 \times a_2 = \sqrt{a} a_3 \quad \text{on the one hand,} \quad (4.4.39)$$

$$a_1 \times a_2 = (a_1 \cdot a_h^{i+})(a_2 \cdot a_h^{j+}) a_{hi}^+ \times a_{hj}^+ \quad \text{on the other hand.} \quad (4.4.40)$$

Using relations (2.1.7) (4.4.34) (4.4.38) to (4.4.40) we can show that

$$(a_1 \times a_2)_{h1}^+ = \sqrt{a_{h11}^+} \cos \delta_1 = \sqrt{a_h^{+22+} a_{11} a_{22}} (\cos \beta_1 \cos \gamma_2 - \cos \gamma_1 \cos \beta_2) \quad (4.4.41)$$

$$(a_1 \times a_2)_{h2}^+ = \sqrt{a_{h22}^+} \cos \delta_2 = \sqrt{a_h^{+11+} a_{11} a_{22}} (\cos \alpha_2 \cos \gamma_1 - \cos \alpha_1 \cos \gamma_2) \quad (4.4.42)$$

$$(a_1 \times a_2)_{h3}^+ = \sqrt{a} \cos \delta_3 = \sqrt{a_h^{+11+} a_h^{+22+} a_{11} a_{22}} (\cos \alpha_1 \cos \beta_2 - \cos \alpha_2 \cos \beta_1) \quad (4.4.43)$$

Moreover, the relations (4.4.34) (4.4.38) involve

$$a_3 \cdot a_h^{1+} = \sqrt{a_h^{11+}} \cos \alpha_3 = a_h^{11+} \sqrt{a_{h11}^+} \cos \delta_1 + a_h^{12+} \sqrt{a_{h22}^+} \cos \delta_2 \quad (4.4.44)$$

Using the relations (4.4.36) (4.4.41) to (4.4.44) in connection with the following relations

$$a_{h11}^+ = a_h^{+22+} a_h^{11+}, \quad a_{h22}^+ = a_h^{+11+} a_h^{12+},$$

we derive

$$\left. \begin{aligned} \cos \alpha_1 \cos \beta_2 - \cos \alpha_2 \cos \beta_1 &= \sqrt{\frac{a}{a_h^+}} \frac{\cos \delta_3}{\sqrt{a_h^{11+} a_h^{22+} a_{11} a_{22}}} \\ \sqrt{1-A^2} (\cos \alpha_2 \cos \gamma_1 - \cos \alpha_1 \cos \gamma_2) &= \sqrt{\frac{a}{a_h^+}} \frac{\cos \delta_2}{\sqrt{a_h^{11+} a_h^{22+} a_{11} a_{22}}} \\ \cos \beta_2 \cos \gamma_1 - \cos \beta_1 \cos \gamma_2 - A(\cos \alpha_2 \cos \gamma_1 - \cos \alpha_1 \cos \gamma_2) &= \\ &= -\sqrt{\frac{a}{a_h^+}} \frac{\cos \alpha_3}{\sqrt{a_h^{11+} a_h^{22+} a_{11} a_{22}}} \end{aligned} \right\}$$

Finally, by substitution in the expression (4.4.37) we get :

$$D = \left( \frac{a}{a_h} \right)^{3/2} \cos \delta_3 \{ \cos^2 \delta_3 + B \cos^2 \delta_2 + B \cos^2 \alpha_3 \}.$$

But the three vectors  $\frac{a_h^{1+}}{\sqrt{a_h^{11+}}}$ ,  $\frac{a_{h2}^+}{\sqrt{a_{h22}^+}}$ ,  $a_{h3}^+$  are orthonormal.

Then, the relations (4.4.34) and (4.4.38) involve

$$\cos^2 \alpha_3 + \cos^2 \delta_2 + \cos^2 \delta_3 = 1 ,$$

from which we derive

$$D = \left( \frac{a}{a_h} \right)^{3/2} \cos^3 \delta_3 \{1 + B \tan^2 \delta_3\} . \quad (4.4.45)$$

If we observe that the POISSON coefficient  $\nu$  is always such that  $-1 < \nu < \frac{1}{2}$ , it comes with (4.4.36)

$$D = \left( \frac{a}{a_h} \right)^{3/2} \cos \delta_3 \frac{\cos^2 \delta_3^{-\nu}}{1-\nu} \quad (4.4.46)$$

Then the relation (4.4.46) combined with the Hypothesis 4.4.1 involve  $D > 0$  and the proof of the Lemma 4.4.1 is achieved.

□

Geometrical aspects of the functions of the space  $\tilde{X}_h$  :

By analogy with section 4.2 one can check that if we refer the functions  $\tilde{v}_{hi}$  defined on each triangle  $K \in \mathcal{T}_h$  to the normal to the domain  $\Omega$ , one gets :

(i) for the functions  $\tilde{v}_{h1}$  and  $\tilde{v}_{h2}$  a discontinuous curved faceted surface and, for each facet,  $\tilde{v}_{h\alpha}|_K \in P_3(K)$  ;

(ii) for the function  $\tilde{v}_{h3}$  a discontinuous curved faceted surface but now, for each facet,  $\tilde{v}_{h3}|_K \in P_K$ ,  $P_K$  defined by (4.2.4).

Definition of the space  $\vec{V}_h$  :

By analogy with section 4.2 we shall define the space  $\vec{V}_h$  as the image of the space

$$\vec{V}_h = \{ \vec{v}_h \in \vec{X}_h ; \vec{v}_h|_{\Gamma_0} = \vec{0} ; \partial_{\nu} v_{h3} = 0 \text{ on } \Gamma_0 \} \quad (4.4.47)$$

through the bijection  $F_h^{-1}$ . According to the definition of the space  $\vec{X}_h$ , to get the boundary conditions  $\vec{v}_h|_{\Gamma_0} = \vec{0}$  and  $\partial_{\nu} v_{h3} = 0$  on  $\Gamma_0$ , it suffices to equal to zero the following combinations of degrees of freedom :

$$\vec{v}_h(\Sigma) = \vec{0} \quad (4.4.48)$$

$$v_{h3,\nu}(\Sigma) = 0 \quad (4.4.49)$$

$$(\vec{t} \cdot \vec{E}_1) v_{h\alpha,1}(\Sigma) + (\vec{t} \cdot \vec{E}_2) v_{h\alpha,2}(\Sigma) = 0 \quad (4.4.50)$$

where  $\Sigma$  denotes any vertex of  $\mathcal{T}_h$  which belongs to  $\Gamma_0$ , where  $\vec{E}_1, \vec{E}_2$  denote the reference orthonormal basis of the plane which contains the domain  $\Omega$  and where  $\vec{t}$  denotes a vector which has the direction of  $\Gamma_0$  at vertex  $\Sigma$ . Naturally, if  $\Sigma$  is at a corner of  $\Gamma_0$  the condition (4.4.50) must be satisfied for two different directions  $\vec{t}_1$  and  $\vec{t}_2$ , that means in this case the conditions (4.4.50) must be replaced by  $v_{h\alpha,\beta}(\Sigma) = 0$ .

Then, the consideration of relations (4.4.18) to (4.4.21) shows that the conditions (4.4.48) to (4.4.50) involve

$$\vec{A}(\Sigma) = 0 , \quad B^{\lambda}(\Sigma) = 0$$

and generally,

$$B^3(\Sigma) \neq 0 , \quad C_{\alpha\beta}(\Sigma) \neq 0 .$$



That means that the image of the boundary conditions  $(\vec{v}_h|_{\Gamma_0} = \vec{0}$  and  $\partial_\nu v_{h3} = 0$  on  $\Gamma_0$ ) through the bijection  $F_h^{-1}$  does not lead to simple boundary conditions for the functions of  $\vec{V}_h$ . Thus, we set

$$\vec{V}_h = \{\vec{v}_h | \vec{v}_h = F_h^{-1}(\vec{v}_h) \text{ , } \forall \vec{v}_h \in \vec{V}_h\} \text{ .}$$

□

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